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THE LIFT AND MOMENT ACTING ON A THICK AEROFOIL IN UNSTEADY MOTION

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CONTENTS

A new approach to the problem of calculating the unsteady two-dimensional flow about aerofoils in an incompressible fluid is presented. While the flow is assumed to be inviscid throughout, the potential flow boundary conditions are modified semi-empirically to make some allowance for viscous effects. The method is applicable to thick aerofoils, the only limitation being that the velocities and displacements of the unsteady perturbations about the mean steady motion must be small. The dependent variable of the method is the complex harmonic function $\ln(U\,e^{i\theta}/q)$, where U is the velocity at infinity, and (q, θ) is the velocity vector in polar co-ordinates, while the independent variables are ϕ_0 and ψ_0, the potential and stream functions of the mean steady flow about the aerofoil. The wake is assumed to be a vortex sheet drifting down the trailing edge streamline of the mean steady motion with the (local) velocity of the steady motion. Algebraic results are obtained for the lift and moment by an application of Blasius's theorem. The theory

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is applied to the special case of an aerofoil describing harmonic oscillations which have persisted for an infinite time.

Provided the Reynolds number is sufficiently large for boundary-layer theory to be applicable, viscosity can be regarded as having three main effects on the theoretical potential flow. These are: (i) that it modifies the Joukowski condition that the position of the rear stagnation point is independent of incidence, (ii) that it modifies the velocity distribution of the mean steady flow, particularly near the trailing edge, and (iii) that it contributes a term to the damping which exists even in still air. The author's theory allows for (i) and (ii) but not for (iii).

The theory yields three distinct effects of aerofoil thickness on the classical flat-plate oscillatory derivatives. These are: (i) the reduced frequency ω is replaced by $\delta\omega$, where $2\pi\delta$ is the theoretical lift-slope, (ii) the force and moment derivatives are multiplied by factors of δ, and (iii) the reduction of the wake velocity due to the thickness of the aerofoil changes the flat-plate derivatives by an amount which is quite large at high values of ω.

Tables are given from which it is possible to deduce with little effort the effects of aerofoil thickness and viscosity on the air-load coefficients. The theory yields results in close agreement with some recent Dutch measurements of the air-load coefficients and also with some earlier measurements at the National Physical Laboratory.

Notation

(a) *General*

(x, y) a physical plane fixed in space,

$z = x+\mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$,

(x_0, y_0) a physical plane fixed relative to the aerofoil,

$z_0 = x_0+\mathrm{i}y_0$,

$w = \phi+\mathrm{i}\psi$, the complex stream function of the unsteady flow,

(q, θ) velocity vector in polar co-ordinates,

0 as a suffix to denote values in the steady flow of zero circulation,

a the aerofoil transforms into the slit $\psi_0 = 0$, $-2a \leqslant \phi_0 \leqslant 2a$,

(η, γ) elliptic co-ordinates defined by equation (6),

U velocity at infinity,

f is defined by the equation

$$f = \ln\left(U\frac{\mathrm{d}z}{\mathrm{d}w}\right) = \ln(U/q)+\mathrm{i}\theta, \tag{1}$$

α incidence measured from the no-lift position,

(u, v) velocity components of the aerofoil along the OX and OY axes respectively,

(u_0, v_0) velocity components of the aerofoil along the O_0X_0 and O_0Y_0 axes respectively,

K the jump in value of $\ln(U/q)$ across the wake,

ϕ_s defines the position of the end of the wake,

$-\mu$ the value of η at the end of the wake, $\phi_0 = \phi_s$, $\psi_0 = 0$,

a_n, b_n, c_n defined by equations (51), (52) and (48) respectively,

c, A the chord length and cross-sectional area of the profile respectively.

k the radius of gyration of the profile about the origin in the z_0-plane,

z_c the centroid of the profile in the z_0-plane,

z_a the profile centre defined by equation (38),

$\delta = 4a/Uc$,

D, L, M the drag, lift and moment about the origin of the z_0-plane respectively,

C_L, C_D lift and moment coefficients,

ρ density,
σ_1, σ_2 values of γ at the front and rear stagnation points respectively.

(b) Oscillatory motion

$\nu/2\pi$ the frequency of harmonic oscillations,
ω the reduced frequency, ν/cU,
$\lambda = \frac{1}{2}\omega\delta$, a modified reduced frequency,
$C = A - \mathrm{i}B$, Theodorsen's function, the parameter of which is λ,
x^0, y^0, α^0 the amplitude of harmonic displacements; see equation (64),
ϵ a factor determining the viscous slip at the trailing edge; see equation (65).

1. Introduction

The methods employed hitherto on the theory of unsteady aerofoil motion are: (i) the velocity potential method, (ii) the method of vortices, and (iii) the method of the acceleration potential. An account of these methods and their relative merits has been given by Greidanus & Van Heemert (1948). The present paper introduces another approach to the theory of unsteady aerofoil motion in two dimensions, based on the analytic character of the function f defined by equation (1). The independent variables are ϕ_0 and ψ_0, the potential and stream functions of the steady no-lift flow about the aerofoil. In the plane defined by these variables the boundary conditions are relatively simple—the imaginary part of f, namely, θ, is given on the aerofoil surface, $\psi_0 = 0$, $-2a \leqslant \phi_0 \leqslant 2a$, while the jump, K, in the real part of f, is known, or can be calculated across the wake, $\psi_0 = 0$, $2a \leqslant \phi_0 \leqslant \phi_s$. Thus $f(w_0)$ can be determined quite easily. This calculation is given in the next section.

The treatment of the aerofoil wake given in § 3 differs in two respects from the usual treatment. First, it is not assumed that the vorticity in the wake drifts downstream with constant velocity U. This, the usual approximation, is replaced by the more accurate condition that the vorticity moves downstream with the *local* velocity of the mean steady flow. This means that owing to the aerofoil thickness the vorticity moves away from the aerofoil with a velocity less than U, which tends to the value U with increasing distance downstream. This 'slowing up of the wake' is found to have a significant effect at large values of the frequency parameter. Secondly, the Kelvin circulation theorem is used instead of the usual condition of 'smooth flow' at the trailing edge. The smooth-flow condition, namely, that the flow always leaves the trailing edge tangentially to the aerofoil surface, is only applicable to cusped trailing edges, whereas Kelvin's theorem is quite general and independent of the aerofoil shape. In any case the smooth-flow condition could not be applied in the method of this paper, since the rear stagnation point is not assumed to be fixed exactly at the trailing edge, but is permitted to make very small excursions on either side of the trailing edge during the unsteady motion of the aerofoil.

It is assumed in the paper that most important effects of viscosity can be allowed for in the (inviscid) potential flow theory, by modifying the boundary conditions of the theoretical potential flow. Viscosity is responsible for the following effects, not allowed for in the classical theory of unsteady aerofoil motion: (i) 'viscous slip' at the trailing edge (non-fulfilment of the Joukowski condition), (ii) a modification of the mean steady velocity distribution, especially near the trailing edge, and (iii) 'viscous damping' of the type present in still air. The author's theory allows for (i) by permitting the rear stagnation point to oscillate about the trailing edge in phase with the local relative incidence, and with such an amplitude that the experimental value of the steady lift slope is reproduced by the theory in the limit as the frequency tends to zero. The effect (ii) is allowed for by replacing the theoretical value of z_a, the position of the profile centre† by the experimental value, which can be calculated from an experimental value of the rate of change of the moment coefficient for a known axis position. The author's theory does not allow for (iii), the calculation of which is quite a difficult problem in the theory of viscous flow. W. P. Jones (1948) has developed a thin aerofoil theory allowing for (i) in which the 'equivalent' thin wing distorts in phase with the motion and with such an amplitude distribution that the experimental steady motion characteristics are obtained from the theory in the limit as the frequency tends to zero. Inasmuch as the steady motion characteristics taken from experiment are affected by the aerofoil thickness, Jones's theory also makes partial allowance for the effect of thickness.

Although it will be assumed that the rear stagnation point makes small movements during the course of the unsteady motion, the wake vorticity will be supposed to lie along the fixed trailing edge streamline of the mean steady motion. Even with a fixed rear stagnation point the vorticity would not lie exactly along this streamline, but as demonstrated by Greidanus & Van Heemert (1948), the errors arising from this approximation in the position of the vorticity are negligible. The same type of approximation—that is, an approximation in the *position* of the boundary conditions—is employed on the aerofoil surface. The boundary conditions for the unsteady part of the motion are applied at the *mean* position of the aerofoil surface. This approximation clearly imposes a limit on the amplitude of the unsteady perturbations about the mean motion, and the theory can only be regarded as correct to first order in the amplitudes and velocities of the perturbations.

In § 4 the lift and moment is calculated by means of Blasius's theorem for unsteady flow. An extensive study of the application of this theorem to the unsteady motion of aerofoils has been made recently by Couchet (1949, 1950). He obtains formulae allowing for the effect of aerofoil thickness on the lift and moment, but does not allow in any way for the effects of viscosity. No comparison with experiment is made. For modern aerofoil shapes the viscous effect on the lift and moment is often larger and of opposite sign to the thickness effect, so that a theory allowing for thickness only can be further away from the experimental values than the simple flat-plate theory.

In § 5 the theory is applied to the case of a rigid aerofoil. While the method can be readily extended in an obvious way to the case of a non-rigid aerofoil, e.g. to an aerofoil fitted with a hinged flap, in the interests of brevity the calculation is not given in this paper. The theory given in part I, which applies to any type of unsteady motion, is developed in part II for the special case of harmonic oscillations which have persisted for a relatively long time. For the particular case when the aerofoil degenerates to a flat plate, and the 'viscous slip' at the trailing edge is zero, the theory given in part II correctly reduces to the classical flat-plate theory. The paper concludes with a comparison of theory and experiment, from which it appears that the theory satisfactorily accounts for most of the discrepancies between the classical theory and experimental results.

† Defined on p. 140—the mid-chord point for a flat plate.

PART I. GENERAL THEORY

2. The differential equation and its solution

Since f defined by equation (1) is an analytic function of z, it satisfies Laplace's equation in the z-plane, and in any other plane derived from the z-plane by a conformal transformation. In this section we solve Laplace's equation, with the appropriate boundary conditions, in an auxiliary ζ-plane derived from the z-plane by a conformal transformation. The plane defined by

$$z_0 = (z-z_1)\,e^{i\alpha} \tag{2}$$

and shown in figure 1, has its origin fixed in the aerofoil at O_0. Both z_1, the position of O_0 in the z-plane, and α are functions of the time, t, in fact

$$\frac{dz_1}{dt} = u+iv, \tag{3}$$

$$\frac{d\alpha}{dt} = \dot{\alpha}, \tag{4}$$

where $u+iv$ is the velocity of O_0 in the z-plane and $\dot{\alpha}$ is the (nose-up) angular velocity of the profile.

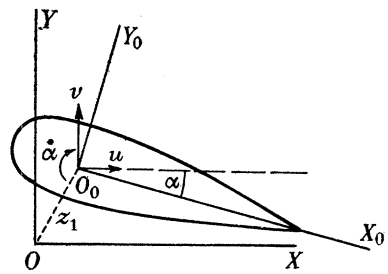

Figure 1

The complex stream function for the steady zero-circulation flow is denoted by $w_0 = \phi_0+i\psi_0$, and the aerofoil is represented in this plane by the slit $\psi_0 = 0$, $-2a \leqslant \phi_0 \leqslant 2a$. The leading edge of the aerofoil, or, more precisely, the front stagnation point for the steady flow of zero circulation, is at $\phi_0 = -2a$, while the rear stagnation point is at $\phi_0 = 2a$.

The w_0-plane is rigidly embedded in the z_0-plane, the O_0X_0 axis of which is taken to be parallel to the direction of the steady flow at infinity. Thus, as shown in figure 4, θ_0 is the angle between the steady-flow direction and the O_0X_0 axis. Similarly, θ is taken to be the angle between the unsteady-flow direction and the OX axis. This particular selection of the O_0X_0 and OX axes makes f (see equation (1)) and f_0, where

$$f_0 = \ln\left(U\frac{dz_0}{dw_0}\right), \tag{5}$$

vanish at infinity.

It is convenient to introduce an auxiliary plane, $\zeta = \eta+i\gamma$, defined by

$$w_0 = -2a\cosh\zeta, \tag{6}$$

so that on the aerofoil surface $\eta = 0$, and

$$\phi_0 = -2a\cos\gamma. \tag{7}$$

The conformal transformation (6) is illustrated in figure 2. If m is a positive or negative integer then the whole of the w_0-plane is mapped into each of the semi-infinite strips

$$-\infty \leqslant \eta \leqslant 0, \quad (2m-1)\,\pi \leqslant \gamma \leqslant (2m+1)\,\pi. \tag{8}$$

The aerofoil is repeated periodically on $\eta = 0$, while the trailing edge streamline lies on the lines $\gamma = (2m+1)\,\pi$.

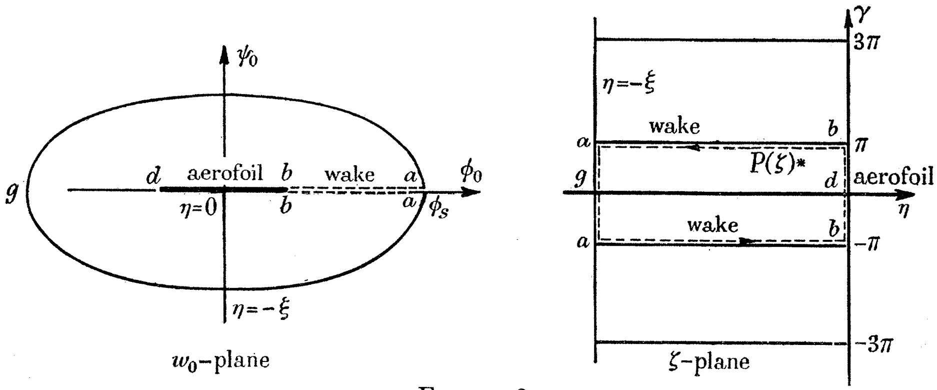

FIGURE 2

The wake produced by the unsteady motion of the aerofoil will be assumed to be a vortex sheet lying on the trailing edge streamline of the steady flow, i.e. on $\gamma = \pi$, $-\mu \leqslant \eta \leqslant 0$. The end of the wake, $\eta = -\mu$, drifts downstream with the velocity of the steady flow, so that after an infinite time $\mu = \infty$. Since the tangential velocity is discontinuous across the vortex sheet, f has a jump in the value of its real part on the lines $\gamma = (2m+1)\,\pi$, when $-\mu \leqslant \eta \leqslant 0$.

Returning now to the idea expressed in the first paragraph of this section, we have that $f(\zeta)$ is an analytic function, since equations (2), (6) and $w_0 = w_0(z_0)$ are all conformal transformations. Besides the discontinuities in f on $\gamma = (2m+1)\,\pi$ mentioned above, $f(\zeta)$ may also have logarithmic singularities on the aerofoil surface, $\eta = 0$, due to stagnation points or sharp corners. However, *within* the contours given by (8), $f(\zeta)$ is a well-behaved analytic function which plainly satisfies

$$f(\zeta) = f(\zeta + 2\mathrm{i}m\pi). \tag{9}$$

The fundamental theorem of this paper will now be stated and proved.

THEOREM. *If $\theta(\gamma^*)$ is the value of θ on the aerofoil surface and $K(\eta^*)$ is the jump in the value of* $\ln\,(U/q)$ *across the wake, $\gamma = \pi$, $-\mu \leqslant \eta^* \leqslant 0$, then at any point $P(\zeta)$, f is given by*

$$f(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta(\gamma^*)\cot\tfrac{1}{2}(\gamma^* + \mathrm{i}\zeta)\,\mathrm{d}\gamma^* + \frac{\mathrm{i}\sinh\zeta}{2\pi}\int_{-\mu}^{0}\frac{K(\eta^*)\,\mathrm{d}\eta^*}{\cosh\zeta + \cosh\eta^*}. \tag{10}$$

(In general θ and K are functions of time.)

Proof. Now f is an analytic function in the regions

$$-\xi \leqslant \eta \leqslant 0, \quad (2m-1)\,\pi < \gamma < (2m+1)\,\pi \tag{11}$$

($m = 0, \pm 1, \pm 2, \ldots, \pm\infty$). Thus from Cauchy's *theorem* in the case $m \neq 0$, and Cauchy's *integral* in the case $m = 0$, it follows with the aid of equation (9) that

$$\begin{matrix} m \neq 0 & 0 \\ m = 0 & f(\zeta) \end{matrix} = \frac{1}{2\pi\mathrm{i}}\left\{\int_{-\pi}^{\pi}\left(\frac{f(0,\gamma)}{\mathrm{i}\gamma + 2m\mathrm{i}\pi - \zeta} - \frac{f(-\xi,\gamma)}{-\xi + \mathrm{i}\gamma + 2m\mathrm{i}\pi - \zeta}\right)\mathrm{i}\,\mathrm{d}\gamma \right.$$
$$\left. + \int_{-\xi}^{0}\left(\frac{f(\eta,\pi)}{\eta + 2m\mathrm{i}\pi + \mathrm{i}\pi - \zeta} - \frac{f(\eta,-\pi)}{\eta + 2m\mathrm{i}\pi - \mathrm{i}\pi - \zeta}\right)\mathrm{d}\eta\right\}, \tag{12}$$

where $f(\eta, \pm\pi) = \lim_{\epsilon=0} f(\eta, \pi \pm \epsilon)$, and the point $P(\zeta)$ has been assumed to lie within $-\xi \leqslant \eta \leqslant 0$, $-\pi < \gamma < \pi$, without loss of generality. The possible logarithmic singularities in f on $\eta = 0$ make no contribution to the contour integrals, as is easily verified by first indenting the contours with semicircles about these singularities, and then allowing the radii of the indentations to tend to zero.

From (6), $\lim_{\xi=\infty} f(-\xi, \gamma) = \lim_{w_0=\infty} f(w_0) = \lim_{z_0=\infty} f(z_0) = 0$; also $\lim_{M=\infty} \sum_{m=-M}^{M} (a+bm)^{-1} = \frac{\pi}{b}\cot\frac{\pi a}{b}$, and so applying these two limits to the equation obtained by summing equations (12) from $m = -M$ to $m = M$, we find

$$f(\zeta) = \frac{1}{4\pi \mathrm{i}}\int_{-\pi}^{\pi} f(0,\gamma)\cot\tfrac{1}{2}(\gamma+\mathrm{i}\zeta)\,\mathrm{d}\gamma + \frac{1}{4\pi\mathrm{i}}\int_{-\infty}^{0} [f]\tanh\tfrac{1}{2}(\eta-\zeta)\,\mathrm{d}\eta, \tag{13}$$

where $[f] = f(\eta, \pi) - f(\eta, -\pi)$.

If ζ is replaced by $-\bar{\zeta}$ in equation (13) (the bar denoting 'conjugate') then, since $P(-\bar{\zeta})$ is a point outside *all* the regions defined by (11), the left-hand side of the equation will vanish. The conjugate to the equation just described is

$$O = \frac{1}{4\pi\mathrm{i}}\int_{-\pi}^{\pi} \bar{f}(0,\gamma)\cot\tfrac{1}{2}(\gamma+\mathrm{i}\zeta)\,\mathrm{d}\gamma + \frac{1}{4\pi\mathrm{i}}\int_{-\infty}^{0} [\bar{f}]\tanh\tfrac{1}{2}(\eta+\zeta)\,\mathrm{d}\eta. \tag{14}$$

Since θ is continuous across the wake,

$$[f] = [\bar{f}] = \ln(U/q) \equiv K, \tag{15}$$

say, and $K = 0$ when $\eta < -\mu$; also

$$f(0,\gamma) - \bar{f}(0,\gamma) = 2\mathrm{i}\theta(0,\gamma).$$

The theorem now follows immediately by subtraction of equation (14) from (13).

In steady motion $K = 0$. In particular, when the aerofoil is at zero circulation, equation (10) reduces to

$$f_0 \equiv \ln\left(U\frac{\mathrm{d}z_0}{\mathrm{d}w_0}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta_0(\gamma^*)\cot\tfrac{1}{2}(\gamma^*+\mathrm{i}\zeta)\,\mathrm{d}\gamma^*, \tag{16}$$

where θ_0 is the steady part of θ for zero circulation.

3. Conditions in the wake

An integral equation for the function $K(\eta^*)$ of equation (10) can be established with the aid of the following fundamental laws:

(*a*) *the Kelvin circulation theorem*, which states that the circulation or potential jump, $[\phi]_C$, about any circuit C moving with the fluid, is constant, and

(*b*) *the Helmholtz law on the persistence of vorticity*, which can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}[q] = 0, \tag{17}$$

where $[q]$ is the jump in the value of q across the wake (equal to the surface density of vorticity), and $\mathrm{d}/\mathrm{d}t$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \frac{\partial s}{\partial t}\frac{\partial \phi_0}{\partial s}\frac{\partial}{\partial \phi_0},$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + q_0^2\frac{\partial}{\partial \phi_0}, \tag{18}$$

q_0 being the steady velocity along the trailing edge streamline.

In applying Kelvin's theorem we shall suppose that C is any contour completely enclosing the aerofoil and the (finite) wake, then, if Γ is the constant circulation about the aerofoil before the unsteady motions starts, the theorem states

$$[\phi]_C = \Gamma.$$

Closure of the aerofoil clearly requires that $[\psi]_C = 0$, and hence

$$[w]_C = \Gamma. \tag{19}$$

A convenient form of this equation can be obtained by considering the form w takes in the neighbourhood of infinity, i.e. near $\zeta = -\infty$. From equation (6) we find that

$$e^{\zeta} = -\left(\frac{a}{w_0}\right)\left\{1+\left(\frac{a}{w_0}\right)^2+2\left(\frac{a}{w_0}\right)^3+O\left(\frac{a}{w_0}\right)^4\right\}. \tag{20}$$

If equation (10) is now expressed as a power series in e^{ζ}, and then transformed by (20) into a power series in (a/w_0), it will be found that

$$\begin{aligned} f = \ln\left(U\frac{dz}{dw}\right) &= \frac{i}{2\pi}\left(\int_{-\pi}^{\pi}\theta\, d\gamma - \int_{-\mu}^{0} K\, d\eta\right) - \frac{a}{w_0}\frac{i}{\pi}\left(\int_{-\pi}^{\pi}\theta\, e^{-i\gamma}\, d\gamma + \int_{-\mu}^{0} K\cosh\eta\, d\eta\right) \\ &\quad + \left(\frac{a}{w_0}\right)^2 \frac{i}{\pi}\left(\int_{-\pi}^{\pi}\theta\, e^{-2i\gamma}\, d\gamma - \int_{-\mu}^{0} K\cosh 2\eta\, d\eta\right) \\ &\quad - \left(\frac{a}{w_0}\right)^3 \frac{i}{\pi}\left(\int_{-\pi}^{\pi}\theta(e^{-i\gamma}+e^{-3i\gamma})\, d\gamma + \int_{-\mu}^{0} K(\cosh\eta+\cosh 3\eta)\, d\eta\right) + O\left(\frac{a}{w_0}\right)^4. \end{aligned} \tag{21}$$

For steady motion and zero circulation this equation becomes

$$\begin{aligned} f_0 \equiv \ln\left(U\frac{dz_0}{dw_0}\right) &= \frac{i}{2\pi}\int_{-\pi}^{\pi}\theta_0\, d\gamma - \left(\frac{a}{w_0}\right)\frac{i}{\pi}\int_{-\pi}^{\pi}\theta_0\, e^{-i\gamma}\, d\gamma + \left(\frac{a}{w_0}\right)^2\frac{i}{\pi}\int_{-\pi}^{\pi}\theta_0\, e^{-2i\gamma}\, d\gamma \\ &\quad - \left(\frac{a}{w_0}\right)^3\frac{i}{\pi}\int_{-\pi}^{\pi}\theta_0(e^{-i\gamma}+e^{-3i\gamma})\, d\gamma + O\left(\frac{a}{w_0}\right)^4. \end{aligned} \tag{22}$$

Thus, since $\lim_{w_0=\infty} f_0 = 0$, it follows from this equation that

$$\int_{-\pi}^{\pi}\theta_0\, d\gamma = 0. \tag{23}$$

Further, since by definition $[w_0]_C = 0$, a simple application of the theorem of residues to the expansion for f_0 yields

$$\int_{-\pi}^{\pi}\theta_0\, e^{-i\gamma}\, d\gamma = 0. \tag{24}$$

Hence (22) can be written in the form

$$U\frac{dz_0}{dw_0} = 1 + B_0\left(\frac{a}{w_0}\right)^2 + C_0\left(\frac{a}{w_0}\right)^3 + O\left(\frac{a}{w_0}\right)^4, \tag{25}$$

where
$$B_0 = \frac{i}{\pi}\int_{-\pi}^{\pi}\theta_0\, e^{-2i\gamma}\, d\gamma, \tag{26}$$

and
$$C_0 = -\frac{i}{\pi}\int_{-\pi}^{\pi}\theta_0\, e^{-3i\gamma}\, d\gamma. \tag{27}$$

Now $\lim_{w_0=\infty} f$ also vanishes, and so from equation (21) it follows immediately that

$$\int_{-\pi}^{\pi} \theta\,\mathrm{d}\gamma - \int_{-\mu}^{0} K\,\mathrm{d}\eta = 0. \tag{28}$$

Further, from equations (19), (21) and (25) it follows by the theorem of residues that

$$\Gamma = -2a\frac{\mathrm{d}z}{\mathrm{d}z_0}\left\{\int_{-\pi}^{\pi} \theta\,\mathrm{e}^{-\mathrm{i}\gamma}\,\mathrm{d}\gamma + \int_{-\mu}^{0} K\cosh\eta\,\mathrm{d}\eta\right\}.$$

In this paper we shall only be concerned with the case $\Gamma = 0$, i.e. the aerofoil starts the unsteady motion from the position of zero circulation. When Γ is not zero the algebra is more complicated but no new principles are involved. Thus

$$\int_{-\pi}^{\pi} \theta\,\mathrm{e}^{-\mathrm{i}\gamma}\,\mathrm{d}\gamma + \int_{-\mu}^{0} K\cosh\eta\,\mathrm{d}\eta = 0. \tag{29}$$

With the aid of (28) and (29) the expansion (21) can now be written

$$U\frac{\mathrm{d}z}{\mathrm{d}w} = 1 + B\left(\frac{a}{w_0}\right)^2 + C\left(\frac{a}{w_0}\right)^3 + O\left(\frac{a}{w_0}\right)^4, \tag{30}$$

where
$$B = \frac{\mathrm{i}}{\pi}\int_{-\pi}^{\pi} \theta\,\mathrm{e}^{-2\mathrm{i}\gamma}\,\mathrm{d}\gamma - \frac{\mathrm{i}}{\pi}\int_{-\mu}^{0} K\cosh 2\eta\,\mathrm{d}\eta, \tag{31}$$

and
$$C = -\frac{\mathrm{i}}{\pi}\int_{-\pi}^{\pi} \theta\,\mathrm{e}^{-3\mathrm{i}\gamma}\,\mathrm{d}\gamma - \frac{\mathrm{i}}{\pi}\int_{-\mu}^{0} K\cosh 3\eta\,\mathrm{d}\eta. \tag{32}$$

In the application of Helmholtz's law the assumption will be made that $q - q_0$ is small in the wake, but it will not be assumed, as is usual, that $q_0 = U$. Thus

$$K \equiv [\ln(U/q)] = -\left[\ln\left(1 + \frac{q}{q_0} - 1\right)\right] \approx -\left[\frac{q}{q_0} - 1\right] = -\frac{1}{q_0}[q];$$

hence from (17) and (18)
$$\frac{\partial K}{\partial t} + q_0^2\frac{\partial K}{\partial \phi_0} + q_0 K\frac{\partial q_0}{\partial \phi_0} = 0. \tag{33}$$

From (25) the value of q_0 is

$$q_0 = U\left\{1 - a_1\left(\frac{a}{\phi_0}\right)^2 + 2a_2\left(\frac{a}{\phi_0}\right)^3 + O\left(\frac{a}{\phi_0}\right)^4\right\}, \tag{34}$$

where (see appendix 1) a_1 and $-2a_2$ are the real parts of B_0 and C_0 respectively. As shown in appendix 1, a_1 depends on the aerofoil thickness, while a_2 depends on the distance between the profile centre and the profile centroid. Equation (34) gives the value of q_0 in the wake at large values of ϕ_0—the error term increasing as the trailing edge ($\phi_0 = 2a$) is approached. As boundary-layer separation probably introduces a term of order $(a/\phi_0)^4$ or even higher, there is little point in considering more terms in the expansion of q_0. Thus from (33) and (34)

$$\frac{\partial K}{\partial t} + U^2\left\{1 - 2a_1\left(\frac{a}{\phi_0}\right)^2 + 4a_2\left(\frac{a}{\phi_0}\right)^3\right\}\frac{\partial K}{\partial \phi_0} + \frac{2U^2 a_1}{a}\left(\frac{a}{\phi_0}\right)^3 K = 0, \tag{35}$$

where terms $O(a/\phi_0)^4$ have been omitted.

Summarizing these results we have that K must satisfy the integral equations (28) and (29) and the differential equation (35).

4. Calculation of the lift and moment

The lift and moment on a two-dimensional aerofoil in unsteady incompressible flow can be calculated fairly simply from Blassius's theorem. This method has the advantage that surface integrals on the aerofoil can be replaced by integrals around any contour enclosing the aerofoil, and hence it is only necessary to know the forms of the integrands near infinity. For example, in figure 3, integrals around the aerofoil surface, C_1, can be replaced by integrals around contours C_2 and C. The integral over the wake vanishes (in all the cases in which we are interested) owing to the continuity of pressure through the wake, and it only remains to calculate the contribution from the contour C.

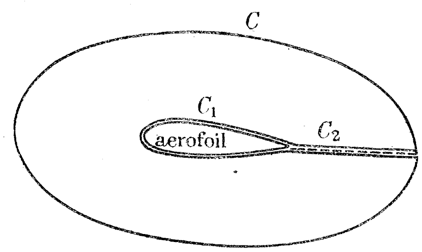

Figure 3

Blasius's theorem has been applied to the problem of the oscillating aerofoil by Sedov (see Nekrasov 1948), and more recently by Couchet (1949, 1950). Sedov's application was to the linearized theory of oscillating aerofoils, whereas Couchet dealt with the thick aerofoil. Apart from the use of Blasius's theorem the author's method is different from Couchet's.

In order to apply Blasius's theorem it is necessary to have the following expansions:

$$Uz_0 = Uz_a + w_0\left\{1 - B_0\left(\frac{a}{w_0}\right)^2 - \tfrac{1}{2}C_0\left(\frac{a}{w_0}\right)^3 + O\left(\frac{a}{w_0}\right)^4\right\}, \tag{36}$$

and
$$\frac{dw}{dw_0} = e^{-i\alpha}\left\{1 + (B_0 - B)\left(\frac{a}{w_0}\right)^2 + (C_0 - C)\left(\frac{a}{w_0}\right)^3 + O\left(\frac{a}{w_0}\right)^4\right\}. \tag{37}$$

The first equation is obtained by integrating (25), Uz_a being the constant of integration, while the second follows from equations (2), (25) and (30). The constant z_a, termed the 'centre of the profile' (see Milne-Thomson 1949), has some importance in aerofoil theory. It can be calculated from

$$z_a = \frac{1}{2\pi}\int_{-\pi}^{\pi} z_0\, d\gamma, \tag{38}$$

which can be proved as follows. From equation (6)

$$\int_{-\pi}^{\pi} z_0\, d\gamma = -i\int_{-\pi}^{\pi} z_0\, d\zeta = i\int_{-\pi}^{\pi} \frac{z_0\, dw_0}{2a\sinh\zeta}.$$

Thus if C is any contour enclosing the aerofoil

$$\int_{-\pi}^{\pi} z_0\, d\gamma = -i\int_C \frac{z_0\, dw_0}{2a\sinh\zeta},$$

where the change in sign is due to the integral on the aerofoil surface being taken in a clockwise direction. Thus from (20),

$$\int_{-\pi}^{\pi} z_0 \,\mathrm{d}\gamma = \mathrm{i}\int_C \frac{z_0}{a}\,\mathrm{e}^{\zeta}\{1+O(\mathrm{e}^{2\zeta})\}\,\mathrm{d}w_0$$

$$= -\mathrm{i}\int_C \frac{z_0}{w_0}\left\{1+O\left(\frac{a}{w_0}\right)^2\right\}\mathrm{d}w_0.$$

Equation (38) follows from this equation by eliminating z_0 using (36), and then applying the residue theorem.

Equation (38) can be written

$$x_a = \frac{1}{2\pi}\int_{-\pi}^{\pi} x_0\,\mathrm{d}\gamma, \quad y_a = \frac{1}{2\pi}\int_{-\pi}^{\pi} y_0\,\mathrm{d}\gamma,$$

which exhibits the profile centre as a kind of centroid. For a symmetrical aerofoil

$$y(\gamma) = -y(-\gamma),$$

therefore $y_a = 0$ and the profile centre lies on the chord line. An approximate equation for x_a is developed in appendix 2. The following remarks may give the reader some idea of the location of the profile centre for any given case. For a body symmetrical about two axes at right angles, such as an ellipse, the profile centre coincides with the geometrical centroid. If the maximum thickness of an aerofoil lies in front of the mid-chord point so will the profile centre (cf. equation (119)). It follows from (36) that if the origin of the z_0-plane is taken to be at the profile centre, then, apart from a scale factor, the z_0 and w_0 planes coincide at infinity.

Blasius's theorem (see Milne-Thomson 1949, § 9·52, where α, $\dot{\alpha}$ and M have the opposite sign) can be stated thus:

If (D, L) is the force acting on the aerofoil per unit span, and M is the nose-up moment about the origin of the z_0-plane, then

$$\mathrm{e}^{-\mathrm{i}\alpha}(D-\mathrm{i}L) = \frac{\mathrm{i}}{2}\rho\int_C\left(\frac{\mathrm{d}w}{\mathrm{d}z_0}\right)^2\mathrm{d}z_0 + \rho\dot{\alpha}\int_C \bar{z}_0\,\mathrm{d}\bar{w} + \mathrm{i}\rho\frac{\partial}{\partial t}\int_C \bar{w}\,\mathrm{d}\bar{z}_0 - \mathrm{i}\rho\Gamma(u_0-\mathrm{i}v_0)$$
$$+\mathrm{i}\rho A\{\dot{\alpha}(u_0-\mathrm{i}v_0+\mathrm{i}\dot{\alpha}\bar{z}_c)-\mathrm{i}(\dot{u}_0-\mathrm{i}\dot{v}_0+\mathrm{i}\bar{z}_c\ddot{\alpha})\}, \qquad (39)$$

and

$$M+\mathrm{i}N = \tfrac{1}{2}\rho\int_C z_0\left(\frac{\mathrm{d}w}{\mathrm{d}z_0}\right)^2\mathrm{d}z_0 - \rho(u_0-\mathrm{i}v_0)\int_C z_0\,\mathrm{d}w + \rho\frac{\partial}{\partial t}\int_C z_0 w\,\mathrm{d}\bar{z}_0$$
$$+\dot{\alpha}\rho A(u_0-\mathrm{i}v_0)\,z_c + A\rho\{3\mathrm{i}z_c(\dot{u}_0-\mathrm{i}\dot{v}_0)-2k^2\ddot{\alpha}\}, \qquad (40)$$

where $\partial/\partial t$ denotes the time rate of change at a point fixed in the z_0-plane and N is merely a dummy symbol. It is apparent from the analytic forms of the integrands in (39) and (40) that only in the case of the last integral in (40) need the contour of integration C be on the aerofoil surface. This last integral requires special attention.

If '$\mathscr{R}$' denotes 'real part of' then

$$\mathscr{R}\int_C z_0 w\,\mathrm{d}\bar{z}_0 = \mathscr{R}\int_C z_0 w\,\mathrm{d}z_0 + 2\int_C (x_0\psi + y_0\phi)\,\mathrm{d}y_0,$$

the last integral of which vanishes in the linearized theory. From (40) it is apparent that only the small time-dependent part of ϕ in $\int_C y_0\phi\,\mathrm{d}y_0$ is important. Thus neglect of this integral would at most result in an error of second order in the thickness effect. Making this approximation, and writing

$$\psi = -v_0 x_0 + u_0 y_0 + \tfrac{1}{2}\dot{\alpha}(x_0^2 + y_0^2)$$

(cf. Milne-Thomson 1949, § 9·40) we find from Green's theorem in two dimensions that

$$\mathscr{R}\int_C z_0 w\,\mathrm{d}\bar{z}_0 = \mathscr{R}\int_C z_0 w\,\mathrm{d}z_0 - 4v_0\iint x_0\,\mathrm{d}x_0\,\mathrm{d}y_0 + 2u_0\iint y_0\,\mathrm{d}x_0\,\mathrm{d}y_0 + \dot{\alpha}\iint(3x_0^2 + y_0^2)\,\mathrm{d}x_0\,\mathrm{d}y_0,$$

whence, with neglect of a further second-order term in thickness†

$$\mathscr{R}\rho\frac{\partial}{\partial t}\int_C z_0 w\,\mathrm{d}\bar{z}_0 = \mathscr{R}\rho\frac{\partial}{\partial t}\int_C z_0 w\,\mathrm{d}z_0 - 4\rho x_c A\dot{v}_0 + 2\rho y_c A\dot{u}_0 + 3k^2\rho A\ddot{\alpha}.$$

This equation enables (40) to be written

$$M + \mathrm{i}N = \tfrac{1}{2}\rho\int_C z_0\left(\frac{\mathrm{d}w}{\mathrm{d}z_0}\right)^2\mathrm{d}z_0 - \rho(u_0 - \mathrm{i}v_0)\int_C z_0\,\mathrm{d}w + \rho\frac{\partial}{\partial t}\int_C z_0 w\,\mathrm{d}z_0$$
$$+ \dot{\alpha}\rho A(u_0 - \mathrm{i}v_0)\,z_c + \rho A(k^2\ddot{\alpha} + \mathrm{i}z_c(\dot{u}_0 + \mathrm{i}\dot{v}_0)), \qquad (41)$$

where C can now be any contour enclosing the aerofoil.

There is obviously no ultimate loss in generality in taking the origin of the z_0-plane to be at the profile centre, so that $z_a = 0$, and this is done in the following analysis. The transformation to a general axis position is given in § 12.

The integrals in (39) and (41) can be calculated from the theorem of residues with the aid of the expansions (25), (30), (36) and (37). The results are:

$$\int_C\left(\frac{\mathrm{d}w}{\mathrm{d}z_0}\right)^2\mathrm{d}z_0 = 0, \quad \int_C \bar{z}_0\,\mathrm{d}\bar{w} = -\int_C \bar{w}\,\mathrm{d}\bar{z}_0 = \frac{2\pi\mathrm{i}a^2}{U}\bar{B}\,\mathrm{e}^{\mathrm{i}\alpha},$$

$$\int_C z_0\left(\frac{\mathrm{d}w}{\mathrm{d}z_0}\right)^2\mathrm{d}z_0 = -4\pi\mathrm{i}a^2\,\mathrm{e}^{-2\mathrm{i}\alpha}B, \quad \int_C z_0\,\mathrm{d}w = -\frac{2\pi\mathrm{i}a^2}{U}B\,\mathrm{e}^{-\mathrm{i}\alpha},$$

and
$$\int_C z_0 w\,\mathrm{d}z_0 = \frac{\pi\mathrm{i}a^3}{U^2}C\,\mathrm{e}^{-\mathrm{i}\alpha},$$

which enable (39) and (41) to be written

$$D + \mathrm{i}L = \frac{2\pi\rho a^2}{U}\frac{\partial}{\partial t}(B\,\mathrm{e}^{-2\mathrm{i}\alpha}) + \rho A\{(\dot{u} + \mathrm{i}\dot{v}) - z_c(\dot{\alpha}^2 + \mathrm{i}\ddot{\alpha})\,\mathrm{e}^{-\mathrm{i}\alpha}\}, \qquad (42)$$

$$M + \mathrm{i}N = -2\pi\mathrm{i}\rho a^2\,\mathrm{e}^{-2\mathrm{i}\alpha}B\left(1 - \frac{u - \mathrm{i}v}{U}\right) + \frac{\rho\mathrm{i}\pi a^3}{U^2}\mathrm{e}^{-\mathrm{i}\alpha}\left(\frac{\partial C}{\partial t} - \mathrm{i}\dot{\alpha}C\right)$$
$$-\mathrm{i}\rho A z_c\{2\dot{\alpha}(u\sin\alpha + v\cos\alpha) - (\dot{u} + \mathrm{i}\dot{v})\,\mathrm{e}^{\mathrm{i}\alpha}\} + \rho A k^2\ddot{\alpha}, \qquad (43)$$

in which Γ has been made equal to zero, and u_0, v_0 have been transformed into u, v (see Notation, $(u_0 + \mathrm{i}v_0)\,\mathrm{e}^{-\mathrm{i}\alpha} = u + \mathrm{i}v$).

It only remains now to calculate the values of B and C for any particular example. In the next section B and C are evaluated for the important case of a rigid aerofoil.

† The approximations introduced here can be avoided at the cost of considerable algebra.

5. The case of a rigid aerofoil

Figure 4 shows details of the motion of the aerofoil. $P(x_0, y_0)$ is a point on the aerofoil surface, and PQ and PR are parallel to OX and O_0X_0 respectively. The velocity at P can be resolved into components normal and tangential to the aerofoil surface. The normal component must be equal to the velocity (n) of the surface in the direction of its normal. The origin O_0 has perturbation velocities $u\cos\alpha - v\sin\alpha$, and $u\sin\alpha + v\cos\alpha$ in the directions of O_0X_0 and O_0Y_0 respectively, and hence the normal velocity of the surface at P due to this motion is

$$n = (u\sin\alpha + v\cos\alpha - \dot{\alpha}x_0)\cos\theta_0 - (u\cos\alpha - v\sin\alpha + \dot{\alpha}y_0)\sin\theta_0.$$

The tangential velocity at P is of the form $q_0+\epsilon$, where ϵ is the small perturbation in q_0 due to the unsteady motion. It follows from the figure that

$$\theta = \theta_0 + \tan^{-1}\left(\frac{n}{q_0+\epsilon}\right) - \alpha,$$

and hence to first order in perturbation velocities and displacements,

$$\theta = \theta_0 + \frac{v}{q_0}\cos\theta_0 - \frac{u}{q_0}\sin\theta_0 - \frac{\dot{\alpha}}{q_0}(x_0\cos\theta_0 + y_0\sin\theta_0) - \alpha. \tag{44}$$

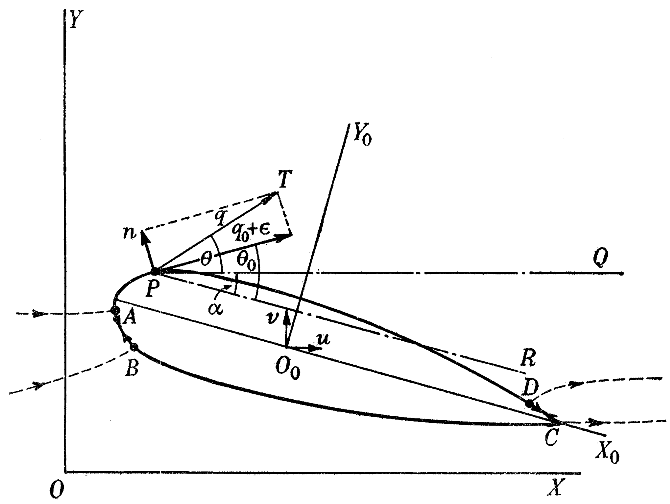

Figure 4

This equation breaks down when q_0 is very small, that is, in the immediate neighbourhood of the stagnation points. In these neighbourhoods, however, by far the more important contribution to θ arises from the movement of the stagnation points. Consider, for instance, the front stagnation point. If A is its position in steady flow (at $\gamma = 0$) and B is its position at any instant in the unsteady flow, then in the interval AB the value of θ given by (44) will be increased by π owing to the reversal in flow direction. Thus (44) is modified to read

$$\theta = \theta_0 + \frac{v}{q_0}\cos\theta_0 - \frac{u}{q_0}\sin\theta_0 - \frac{\dot{\alpha}}{q_0}(x_0\cos\theta_0 + y_0\sin\theta_0) - \alpha \begin{cases} +\pi & (-\sigma_1 \leqslant \gamma \leqslant 0), \\ & (-\pi \leqslant \gamma \leqslant \pi), \\ +\pi & (\pi - \sigma_2 \leqslant \gamma \leqslant \pi), \end{cases} \tag{45}$$

where $-\sigma_1$ and $-\sigma_2$ are the displacements in γ of the front and rear stagnation points respectively. An exact treatment in the neighbourhood of the stagnation points would be very difficult, if not impossible, as this would require knowing ϵ from the start, but as the range over which the errors in (45) are relatively large, is only of the same order as the perturbation magnitudes themselves, these errors can only contribute second-order terms to the integrals giving the lift and moment.

From (23), (24), (28), (29) and (45) it follows that

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{v}{q_0}\cos\theta_0 - \frac{u}{q_0}\sin\theta_0 - \frac{\alpha}{q_0}(x_0\cos\theta_0 + y_0\sin\theta_0)\right)\mathrm{d}\gamma + \sigma_1 + \sigma_2 - 2\alpha, \tag{46}$$

$$c_1 = -\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{v}{q_0}\cos\theta_0 - \frac{u}{q_0}\sin\theta_0 - \frac{\dot{\alpha}}{q_0}(x_0\cos\theta_0 + y_0\sin\theta_0)\right)\cos\gamma\,\mathrm{d}\gamma - \sin\sigma_1 + \sin\sigma_2, \tag{47}$$

where the coefficients c_n $(n = 0, 1)$ are defined by

$$c_n \equiv \frac{1}{\pi}\int_{-\mu}^{0} K\cosh n\eta\,\mathrm{d}\eta. \tag{48}$$

Similarly (31) and (32) become

$$B = B_0 + \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{v}{q_0}\cos\theta_0 - \frac{u}{q_0}\sin\theta_0 - \frac{\dot{\alpha}}{q_0}(x_0\cos\theta_0 + y_0\sin\theta_0)\right) \times (\sin 2\gamma + \mathrm{i}\cos 2\gamma)\,\mathrm{d}\gamma - \sin^2\sigma_1 - \sin^2\sigma_2 + \frac{\mathrm{i}}{2}\sin 2\sigma_1 + \frac{\mathrm{i}}{2}\sin 2\sigma_2 - \mathrm{i}c_2, \tag{49}$$

and
$$C = C_0 - \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{v}{q_0}\cos\theta_0 - \frac{u}{q_0}\sin\theta_0 - \frac{\dot{\alpha}}{q_0}(x_0\cos\theta_0 + y_0\sin\theta_0)\right)(\sin 3\gamma + \mathrm{i}\cos 3\gamma)\,\mathrm{d}\gamma - \tfrac{1}{3}(\cos 3\sigma_1 - 1) + \tfrac{1}{3}(\cos 3\sigma_2 - 1) - \frac{\mathrm{i}}{3}\sin 3\sigma_1 + \frac{\mathrm{i}}{3}\sin 3\sigma_2 - \mathrm{i}c_3, \tag{50}$$

where c_2 and c_3 are defined by equation (49).

In appendix 1 it is shown that the integrals in equations (26), (27), (46), (47), (49) and (50) can be reduced to simple forms in terms of the non-dimensional aerofoil parameters defined by

$$a_n \equiv \frac{U}{a\pi}\int_{-\pi}^{\pi} y_0 \sin n\gamma\,\mathrm{d}\gamma, \tag{51}$$

$$b_n \equiv \frac{U}{a\pi}\int_{-\pi}^{\pi} y_0 \cos n\gamma\,\mathrm{d}\gamma. \tag{52}$$

It is shown in appendix 1 that
$$a_1 \doteqdot \frac{8A}{\pi\delta^2 c^2}, \tag{53}$$

$$a_2 \doteqdot -\frac{4x_c}{\delta c}a_1, \tag{54}$$

and
$$a_3 \doteqdot \left(\frac{16k^2}{\delta^2 c^2} - 1\right)a_1, \tag{55}$$

k now being the radius of gyration of the profile about the profile centre (the origin). For a symmetrical aerofoil $b_n = 0$.

The non-dimensional number δ appearing in these equations is the ratio $4a/Uc$, that is, the ratio of the 'chord length' in the w_0-plane to the real chord length multiplied by U.

It occurs throughout the following theory, and has the physical significance of being the theoretical value of the lift slope divided by 2π (cf. equation (87)). A fair approximation to δ for conventional aerofoils is $\delta = 1+0{\cdot}8t/c$, where t is the maximum thickness. An exact equation for δ is given in appendix 2.

In terms of the parameters a_n and b_n equations (46), (47), (49) and (50) become

$$\left.\begin{aligned} c_0 &= \frac{2v}{U}-2\alpha+\sigma_1+\sigma_2,\\ c_1 &= -\frac{2\dot{\alpha}a}{U^2}-\sin\sigma_1+\sin\sigma_2,\end{aligned}\right\} \tag{56}$$

$$B = a_1\left(1-\frac{u}{U}\right)-\frac{vb_1}{U}-\frac{\dot{\alpha}ab_2}{U^2}-\sin^2\sigma_1-\sin^2\sigma_2$$
$$+\mathrm{i}\left\{b_1\left(1-\frac{u}{U}\right)+\frac{va_1}{U}+\frac{\dot{\alpha}aa_2}{U^2}+\tfrac{1}{2}\sin 2\sigma_1+\tfrac{1}{2}\sin 2\sigma_2-c_2\right\},$$

and $$C = -2\left\{a_2\left(1-\frac{u}{U}\right)-\frac{v}{U}b_2-\frac{\dot{\alpha}a}{U^2}(b_1+b_3-a_1b_1)\right\}-\tfrac{1}{3}(\cos 3\sigma_1-1)+\tfrac{1}{3}(\cos 3\sigma_2-1)$$
$$-2\mathrm{i}\left\{b_2\left(1-\frac{u}{U}\right)+\frac{v}{U}a_2+\frac{\dot{\alpha}a}{U^2}(2a_1+2a_3-a_1^2+b)_1^2\right\}-\tfrac{1}{3}\sin 3\sigma_1+\tfrac{1}{3}\sin 3\sigma_2-c_3.$$

Substituting these results for B and C in equations (42) and (43) and retaining only first-order terms in displacements, velocities, σ_1, σ_2 and the parameters a_n and b_n, we obtain the following equations for the drag, lift and moment:

$$D = \frac{2\pi\rho a^2}{U^2}\left\{2b_1U\dot{\alpha}-a_1\dot{u}-b_1\dot{v}-\frac{ab_2}{U}\ddot{\alpha}\right\}+\rho A(\dot{u}+y_c\ddot{\alpha}), \tag{57}$$

$$L = -\frac{2\pi\rho a^2}{U^2}\left\{2a_1U\dot{\alpha}+b_1\dot{u}-a_1\dot{v}-\frac{aa_2}{U}\ddot{\alpha}+U(\dot{c}_2-\dot{\sigma}_1-\dot{\sigma}_2)\right\}+\rho A(\dot{v}-x_c\ddot{\alpha}), \tag{58}$$

and $$M = -2\pi\rho a^2\left\{2a_1\alpha-b_1+c_2-\sigma_1-\sigma_2+2b_1\frac{u}{U}-2a_1\frac{v}{U}+\frac{\dot{u}ab_2}{U^3}-\frac{\dot{v}aa_2}{U^3}\right.$$
$$\left.-\frac{a}{2U^2}(\dot{\sigma}_1-\dot{\sigma}_2+\dot{c}_3)-\frac{a^2\ddot{\alpha}}{U^4}(a_1+a_3)\right\}+\rho A(k^2\ddot{\alpha}-y_c\dot{u}-x_c\dot{v}). \tag{59}$$

To the same order of accuracy, equations (56) yield

$$\tfrac{1}{2}(c_0+c_1) = \frac{v}{U}-\frac{\dot{\alpha}a}{U^2}-\alpha+\sigma_2, \tag{60}$$

and $$\sigma_1-\sigma_2 = -c_1-\frac{2\dot{\alpha}a}{U^2}. \tag{61}$$

The position of the rear stagnation point must be assigned empirically. When this has been done and the motion of the aerofoil is prescribed, (60) is an integral equation for K which must be solved before D, L and M can be calculated. Equation (61) enables the unsteady motion of the front stagnation point to be studied.

The theory developed so far applies to any type of unsteady motion provided the disturbances are small. In the next part the theory is applied to the special but important case of harmonic oscillations which have persisted for an infinite time.

PART II. HARMONIC OSCILLATIONS

6. The wake

In the case of harmonic oscillations (35) becomes

$$\frac{\partial \tilde{K}}{\partial \phi_0}+\frac{\mathrm{i}\nu}{U^2}\left\{1+2a_1\left(\frac{a}{\phi_0}\right)^2-2\left(2a_2+\frac{\mathrm{i}U^2}{a\nu}a_1\right)\left(\frac{a}{\phi_0}\right)^2\right\}\tilde{K}=0,$$

where $\tilde{K}$ is defined by $$K(\phi_0,t)=\tilde{K}(\phi_0)\,\mathrm{e}^{\mathrm{i}\nu t},$$

and terms $O(a/\phi_0)^4$ have been neglected. Solving this equation for $\tilde{K}$ we find that (ignoring second-order terms in a_1 and a_2 as in the previous section)

$$K=K_0\,\mathrm{e}^{\mathrm{i}\nu t}\,\mathrm{e}^{-\mathrm{i}\lambda\cosh\eta}\,\{1+\tfrac{1}{2}\mathrm{i}\lambda a_1\operatorname{sech}\eta+\tfrac{1}{4}(a_1-\mathrm{i}\lambda a_2)\operatorname{sech}^2\eta\}, \tag{62}$$

where K_0 is a constant of integration,

$$\lambda=\frac{2\nu a}{U^2}=\tfrac{1}{2}\delta\omega, \tag{63}$$

and (cf. equation (6)) ϕ_0 has been replaced by $2a\cosh\eta$. The frequency parameter defined by (63) occurs throughout the following analysis. In the case of a flat plate $\delta=1$, when λ reduces to the frequency parameter usually used in oscillatory theory.

The displacements of the aerofoil are taken to be

$$\left.\begin{aligned} &x_1=x^0\,\mathrm{e}^{\mathrm{i}\nu t}, && y_1=y^0\,\mathrm{e}^{\mathrm{i}\nu t}, && \alpha=\alpha^0\,\mathrm{e}^{\mathrm{i}\nu t},\\ \text{so that}\quad &u=\mathrm{i}\nu x^0\,\mathrm{e}^{\mathrm{i}\nu t}, && v=\mathrm{i}\nu y^0\,\mathrm{e}^{\mathrm{i}\nu t}, && \dot{\alpha}=\mathrm{i}\nu\alpha^0\,\mathrm{e}^{\mathrm{i}\nu t},\\ \text{and}\quad &\dot{u}=-\nu^2x^0\,\mathrm{e}^{\mathrm{i}\nu t}, && \dot{v}=-\nu^2y^0\,\mathrm{e}^{\mathrm{i}\nu t}, && \ddot{\alpha}=-\nu^2\alpha^0\,\mathrm{e}^{\mathrm{i}\nu t}.\end{aligned}\right\} \tag{64}$$

Some assumption is now necessary regarding the movement of the rear stagnation point. In this paper it is assumed that σ_2 is in phase with and proportional in magnitude to the relative incidence at the trailing edge, i.e.

$$\sigma_2=\epsilon\left\{\alpha^0\left(1+\frac{\mathrm{i}\nu}{U}x_t\right)-\frac{\mathrm{i}\nu y^0}{U}\right\}\mathrm{e}^{\mathrm{i}\nu t},$$

where ϵ is a real constant and x_t is the distance between the profile centre and the trailing edge. It is convenient to make a slight approximation in this equation. The profile centre, which is the origin in the (x,y)-plane, is situated approximately at the mid-chord point. Thus $x_t\approx\frac{1}{2}c\approx\frac{1}{2}c\left(\frac{4a}{Uc}\right)$, from equation (116). Hence, with the aid of (63), we can write the equation for σ_2 as

$$\sigma_2=\epsilon\left\{\alpha^0(1+\mathrm{i}\lambda)-\frac{2\mathrm{i}\lambda y^0}{\delta c}\right\}\mathrm{e}^{\mathrm{i}\nu t}. \tag{65}$$

As shown later the value of ϵ can be determined from the experimental value of the steady lift slope, $(\partial C_L/\partial\alpha)_{\lambda=0}$.

Some use will be made of the equations† (Watson 1952)

$$\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda\cosh\eta}\cosh n\eta\,\mathrm{d}\eta=-\frac{\mathrm{i}}{2}\,\mathrm{e}^{-n\pi\mathrm{i}/2}H_n^{(2)}(\lambda), \tag{66}$$

and $$H_{n+1}^{(2)}(\lambda)=\frac{2n}{\lambda}H_n^{(2)}(\lambda)-H_{n-1}^{(2)}(\lambda), \tag{67}$$

† Since λ is real the integral (66) is divergent when $n\geqslant 1$. In this case the Abel limit of the integral should be taken, i.e. λ should be replaced by $\lambda-\mathrm{i}\epsilon$ $(\epsilon>0)$ (a divergent oscillation), then the limit of the integral as $\epsilon\to 0$ taken.

where $H_n^{(2)}(\lambda) = J_n(\lambda) - iY_n(\lambda)$ is the Hankel function of order n. All the Hankel functions of the following theory are for argument λ, which can therefore be omitted from the symbol without ambiguity.

TABLE 1

λ	$I_1^{(2)}$		$I_2^{(2)}$		A_4 coeff. of a_1		A_4 coeff. of a_2	
0	1·00000	—	1·00000	—	0	0i	0	0i
0·02	0·98000	−0·06402i	0·98020	+0·00068i	0·0067	−0·0080i	−0·0002	−0·0001i
0·04	0·96001	−0·11037i	0·96080	+0·00244i	0·0116	−0·0162i	−0·0006	−0·0004i
0·06	0·94002	−0·15004i	0·94180	+0·00505i	0·0149	−0·0239i	−0·0012	−0·0006i
0·08	0·92004	−0·18536i	0·92320	+0·00842i	0·0170	−0·0310i	−0·0019	−0·0008i
0·1	0·90008	−0·21743i	0·90500	+0·01244i	0·0178	−0·0375i	−0·0027	−0·0009i
0·2	0·80067	−0·34571i	0·81997	+0·04098i	0·0084	−0·0581i	−0·0043	+0·0008i
0·3	0·70224	−0·43928i	0·74483	+0·08045i	−0·0156	−0·0575i	+0·0025	+0·0056i
0·4	0·60530	−0·50953i	0·67947	+0·12806i	−0·0459	−0·0359i	0·0217	+0·0112i
0·5	0·51032	−0·56180i	0·62371	+0·18176i	−0·0758	+0·0035i	0·0544	+0·0149i
0·6	0·41776	−0·59927i	0·57733	+0·23993i	−0·1010	+0·0556i	0·0993	+0·0149i
0·7	0·32806	−0·62410i	0·54006	+0·30119i	−0·1201	+0·1146i	0·1540	+0·0112i
0·8	0·24166	−0·63787i	0·51160	+0·36438i	−0·1338	+0·1757i	0·2159	+0·0049i
0·9	0·15893	−0·64184i	0·49161	+0·42844i	−0·1439	+0·2351i	0·2828	−0·0015i
1·0	0·08027	−0·63707i	0·47968	+0·49246i	−0·1527	+0·2908i	0·3529	−0·0054i

λ	A_6 coeff. of a_1		A_6 coeff. of a_2	
0	0	0i	0	0i
0·02	0·0073	−0·0083i	−0·0002	−0·0001i
0·04	0·0135	−0·0172i	−0·0006	−0·0004i
0·06	0·0189	−0·0261i	−0·0013	−0·0008i
0·08	0·0235	−0·0350i	−0·0022	−0·0012i
0·1	0·0275	−0·0439i	−0·0033	−0·0016i
0·2	0·0388	−0·0855i	−0·0094	−0·0029i
0·3	0·0389	−0·1219i	−0·0143	−0·0026i
0·4	0·0297	−0·1523i	−0·0156	−0·0008i
0·5	0·0124	−0·1764i	−0·0122	+0·0027i
0·6	−0·0119	−0·1937i	−0·0030	+0·0077i
0·7	−0·0419	−0·2040i	+0·0123	+0·0142i
0·8	−0·0763	−0·2078i	0·0336	+0·0224i
0·9	−0·1137	−0·2054i	0·0606	+0·0325i
1·0	−0·1532	−0·1978i	0·0927	+0·0449i

Since in the present problem the wake extends to infinity, the value of μ in equation (48) will be infinite. Thus from (48) and (62)

$$c_n = \frac{1}{\pi} K_0 \, e^{i\nu t} \int_0^\infty e^{-i\lambda \cosh \eta} \cosh n\eta \left\{ 1 + \frac{i}{2} \lambda a_1 \operatorname{sech} \eta + \tfrac{1}{4}(a_1 - i\lambda a_2) \operatorname{sech}^2 \eta \right\} d\eta.$$

To integrate this expression it is necessary to evaluate the integrals

$$I_1^{(2)} = \frac{2}{\pi} \int_0^\infty e^{-i\lambda \cosh \eta} \operatorname{sech} \eta \, d\eta, \tag{68}$$

and

$$I_2^{(2)} = -\frac{2i}{\pi} \int_0^\infty e^{-i\lambda \cosh \eta} \operatorname{sech}^2 \eta \, d\eta. \tag{69}$$

From (65) it is readily proved that

$$I_1^{(2)} = 1 - \int_0^\lambda H_0^{(2)}(\lambda)\,d\lambda,$$

and

$$I_2^{(2)} = 1 - \lambda + \int_0^\lambda \int_0^\lambda H_0^{(2)}(\lambda)\,d\lambda\,d\lambda,$$

and these functions are set out in table 1. Equations (66), (68) and (69) enable us to write

$$c_0 = -\frac{i}{2} K_0\, e^{i\nu t}\{H_0^{(2)} - \tfrac{1}{2}\lambda a_1 I_1^{(2)} - \tfrac{1}{4}(a_1 - i\lambda a_2)\, I_2^{(2)}\}, \tag{70}$$

$$c_1 = -\tfrac{1}{2} K_0\, e^{i\nu t}\{H_1^{(2)} - \tfrac{1}{2}\lambda a_1 H_0^{(2)} - \tfrac{1}{4}(a_1 - i\lambda a_2)\, I_1^{(2)}\}, \tag{71}$$

$$c_2 = \frac{i}{2} K_0\, e^{i\nu t}\{H_2^{(2)} - \lambda a_1 (H_1^{(2)} + \tfrac{1}{2} I_1^{(2)}) - \tfrac{1}{2}(a_1 - i\lambda a_2)\,(H_0^{(2)} + \tfrac{1}{2} I_2^{(2)})\}, \tag{72}$$

$$c_3 = \tfrac{1}{2} K_0\, e^{i\nu t}\{H_3^{(2)} - \lambda a_1 (H_2^{(2)} + \tfrac{1}{2} H_0^{(2)}) - \tfrac{1}{4}(a_1 - i\lambda a_2)\,(4H_1^{(2)} + 3I_1^{(2)})\}. \tag{73}$$

We notice here that the terms in a_1 and a_2 appearing in these equations have their origin in equation (34), and thus represent the effect of allowing for the fact that the wake velocity is less than U by an amount depending on the aerofoil thickness and shape. It now follows from (60), (63), (64), (65), (70) and (71) that

$$K_0 = \frac{4\left\{\alpha^0(1 + \tfrac{1}{2} i\lambda) - \dfrac{2i\lambda y^0}{\delta c}\right\}(1 - \epsilon) - 2i\epsilon\lambda\alpha^0}{(H_1^{(2)} + iH_0^{(2)})\,(1 - A_1)}, \tag{74}$$

where

$$A_1 = \tfrac{1}{2}\lambda a_1 \frac{H_0^{(2)} + iI_1^{(2)}}{H_0^{(2)} + iH_1^{(2)}} + \tfrac{1}{4}(a_1 - i\lambda a_2)\,\frac{I_1^{(2)} + iI_2^{(2)}}{H_1^{(2)} + iH_0^{(2)}}. \tag{75}$$

7. The front stagnation point

By definition, the position of the front stagnation point at any instant is $\gamma = \sigma_1$. From (61), (63), (64), (65), (71) and (74)

$$\sigma_1 = 2e^{i\nu t} C\left[\left\{\alpha^0(1 + \tfrac{1}{2} i\lambda) - \frac{2i\lambda y^0}{\delta c}\right\}(1 - \epsilon) - \frac{i}{2}\epsilon\lambda\alpha^0\right]\left(\frac{1 - A_2}{1 - A_1}\right) - i\lambda\alpha^0 e^{i\nu t} + \epsilon\left\{\alpha^0(1 + i\lambda) - \frac{2i\lambda y^0}{\delta c}\right\} e^{i\nu t}, \tag{76}$$

where

$$A_2 = \tfrac{1}{2}\lambda a_1 \frac{H_0^{(2)}}{H_1^{(2)}} + \tfrac{1}{4}(a_1 - i\lambda a_2)\,\frac{I_1^{(2)}}{H_1^{(2)}},$$

and C here denotes Theodorsen's function, $H_1^{(2)}/(H_1^{(2)} + iH_0^{(2)})$.

Now if the aerofoil oscillates about an axis with co-ordinates $(\tilde{x}, \tilde{y})$ instead of the origin (which is the profile centre), then it is a matter of simple kinematics to show that the displacements x^0, y^0 and α^0 in (74) and (76) should be replaced by

$$x^0 - \tilde{y}\alpha^0, \quad y^0 + \tilde{x}\alpha^0, \quad \text{and} \quad \alpha^0 \quad \text{respectively.}$$

Consider the simple case of a flat plate in pitching oscillation about $(\tilde{x}, \tilde{y})$ with no viscous slip at the trailing edge. In this case $A_1 = A_2 = \epsilon = 0$, and $\delta = 1$, and (76) yields

$$\sigma_1 = 2\alpha^0 e^{i\nu t} C\left(1 + \tfrac{1}{2} i\lambda - 2i\lambda \frac{\tilde{x}}{c}\right) - i\lambda\alpha^0 e^{i\nu t}. \tag{77}$$

The profile centre is now at the mid-chord point. Since $\lim_{\lambda\to 0} C = 1$, this equation yields the well-known steady-flow result $\lim_{\lambda\to 0} \sigma_1 = 2\alpha^0$. The amplitude χ and phase τ of $\sigma_1/2\alpha^0$ are shown in figure 5 as functions of ω $(=2\lambda)$ for a number of axis positions. These functions are set out in table 2 for a greater range of ω than shown in the figure.

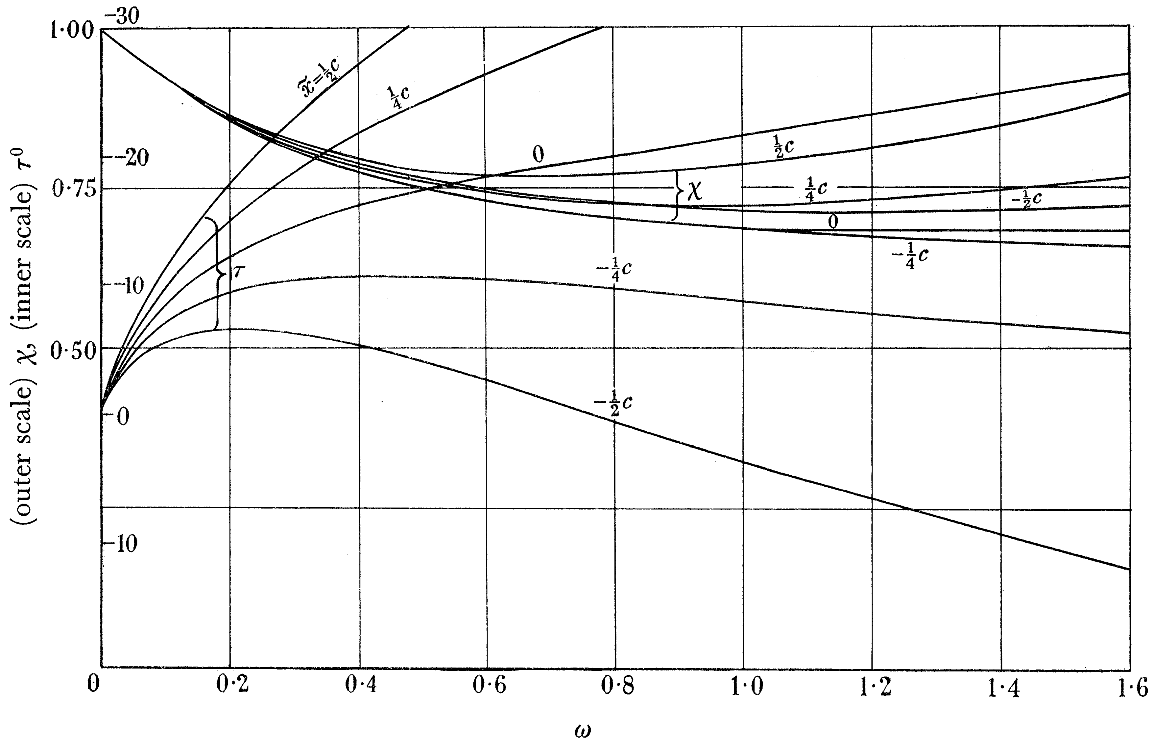

FIGURE 5. The function $\sigma_1/2\alpha^0$ for a flat plate.

In the flow of a real fluid the oscillation of the front stagnation point plays some part in producing an oscillation in the position of boundary-layer separation near the trailing edge. Thus (76) may have some application in the theory of unsteady boundary layers.

TABLE 2. THE FRONT STAGNATION POINT

	$\tilde{x} = \frac{1}{2}c$		$\tilde{x} = \frac{1}{4}c$		$\tilde{x} = 0$		$\tilde{x} = -\frac{1}{4}c$		$\tilde{x} = -\frac{1}{2}c$	
λ	χ	$-\tau^0$	χ	$-\tau^0$	χ	$-\tau^0$	χ	$-\tau^0$	χ	$-\tau^0$
0	1·000	0	1·000	0	1·000	0	1·000	0	1·000	0
0·01	0·984	3·24	0·984	2·96	0·984	2·67	0·984	2·38	0·984	2·10
0·03	0·952	7·67	0·952	6·82	0·952	5·95	0·952	5·09	0·952	4·23
0·05	0·923	11·14	0·922	9·71	0·922	8·29	0·922	6·87	0·923	5·44
0·1	0·865	17·77	0·861	14·96	0·860	12·13	0·860	9·30	0·863	6·49
0·3	0·769	33·92	0·742	26·35	0·729	18·36	0·730	10·24	0·746	2·32
0·5	0·785	44·48	0·720	33·83	0·683	21·56	0·681	8·59	0·713	− 3·84
1·0	0·998	60·65	0·807	48·06	0·676	29·28	0·643	5·44	0·721	−16·84
2·0	1·636	73·77	1·176	64·13	0·788	43·76	0·626	2·91	0·833	−35·28
3·0	2·342	79·02	1·6211	71·80	0·964	54·04	0·626	1·94	1·008	−47·09
4·0	3·070	81·71	2·092	76·07	1·169	61·10	0·625	1·44	1·206	−55·30
5·0	3·806	83·30	2·574	78·75	1·388	66·04	0·625	1·15	1·421	−61·08
10·0	7·528	86·66	5·037	84·30	2·572	77·36	0·625	0·59	2·589	−74·04
∞	∞	90·00	∞	90·00	∞	90·00	0·625	0	∞	−90·00

8. The drag

From (57), (63) and (64)

$$D = \frac{\pi}{2}\rho U^2 \mathrm{e}^{\mathrm{i}\nu t}\left\{a_1 x^0\lambda^2 + b_1 y^0 \lambda^2 + \frac{\delta c\lambda^2}{4} b_2 \alpha^0 + \mathrm{i}\delta\lambda b_1 c\alpha^0\right\} - \rho U^2 \mathrm{e}^{\mathrm{i}\nu t}\frac{\lambda^2 A}{\delta^2 c^2}(x^0 + y_c \alpha^0). \qquad (78)$$

It is of some interest to consider the more general equation (57) for the drag in the case of a symmetrical aerofoil, when $b_n = y_c = 0$. The equation becomes

$$D = \rho\left\{A - \frac{2\pi a^2}{U^2} a_1\right\}\dot{u}.$$

From equations (7), (51) and the symmetry of the aerofoil

$$D = \rho\left\{A - \frac{2}{U}\int_{-2a}^{2a} y_0 \,\mathrm{d}\phi_0\right\}\dot{u},$$

whence the virtual mass, $\tilde{M}$, of the aerofoil is

$$\tilde{M} = \rho A\left\{\frac{2}{AU}\int_{-2a}^{2a} y_0 \,\mathrm{d}\phi_0 - 1\right\}. \qquad (79)$$

As a check on this result consider an elliptic cylinder of semi-axes a and b for which (cf. Milne-Thomson 1949, § 6·32)

$$\phi_0 = -U(a+b)\cos\beta,$$

where β is the eccentric angle of (x_0, y_0), and the flow direction at infinity is parallel to the semi-axis a. For this example, $y_0 = b\sin\beta$, and (79) yields

$$\tilde{M} = \rho A\frac{b}{a} = \rho\pi b^2,$$

in agreement with the standard result (Bairstow 1939).

9. The lift

From (72) and (74)

$$c_2 = \frac{2\mathrm{i}H_2^{(2)}}{H_1^{(2)} + \mathrm{i}H_0^{(2)}}\left[\left\{\alpha^0\left(1 + \frac{\mathrm{i}}{2}\lambda\right) - \frac{2\mathrm{i}\lambda y^0}{\delta c}\right\}(1-\epsilon) - \frac{\mathrm{i}}{2}\epsilon\lambda\alpha^0\right]\left(\frac{1-A_3}{1-A_1}\right)\mathrm{e}^{\mathrm{i}\nu t}, \qquad (80)$$

where $$A_3 = \lambda a_1 \frac{H_1^{(2)} + \frac{1}{2}I_1^{(2)}}{H_2^{(2)}} + \tfrac{1}{2}(a_1 - \mathrm{i}\lambda a_2)\frac{H_0^{(2)} + \frac{1}{2}I_2^{(2)}}{H_2^{(2)}}.$$

By an application of the recurrence relation (67)

$$\frac{\mathrm{i}H_2^{(2)}}{H_1^{(2)} + \mathrm{i}H_0^{(2)}} = \left(\frac{2\mathrm{i}}{\lambda} + 1\right)C - 1,$$

so that from (65), (76) and (80)

$$c_2 - \sigma_1 - \sigma_2 = 2\mathrm{e}^{\mathrm{i}\nu t}\left[\left\{\alpha^0\left(1 + \frac{\mathrm{i}}{2}\lambda\right) - \frac{2\mathrm{i}\lambda y^0}{\delta c}\right\}(1-\epsilon) - \frac{\mathrm{i}}{2}\epsilon\lambda\alpha^0\right]\left(\frac{2\mathrm{i}C}{\lambda} - 1\right)(1 + A_4)$$
$$- 2\epsilon\left\{\alpha^0\left(1 + \frac{\mathrm{i}}{2}\lambda\right) - \frac{2\mathrm{i}\lambda y^0}{\delta c}\right\}\mathrm{e}^{\mathrm{i}\nu t} + \mathrm{i}(1-\epsilon)\lambda\alpha^0\mathrm{e}^{\mathrm{i}\nu t}, \qquad (81)$$

where $$A_4 = \frac{1}{1-A_1}\left\{A_1 - A_3 + \frac{\lambda C(A_2 - A_3)}{2\mathrm{i}C - \lambda}\right\}. \qquad (82)$$

Substituting the derivative of (81) in (58) we find with the aid of (63) and (64) that

$$\frac{L}{\rho U^2 c\,\mathrm{e}^{\mathrm{i}\nu t}} = \left\{\frac{\pi}{2}b_1\lambda^2\right\}\frac{x^0}{c} + \pi\left\{(1-\epsilon)\,(1+A_4)\,(\lambda^2-2\mathrm{i}C\lambda)+\epsilon\lambda^2-\tfrac{1}{2}a_1\lambda^2-\frac{A4\lambda^2}{c^2\pi\delta^2}\right\}\frac{y^0}{c}$$
$$+\pi\delta\left\{\tfrac{1}{2}(1-\epsilon)\,(1+A_4)\,(1+\tfrac{1}{2}\mathrm{i}\lambda)\,(2C+\mathrm{i}\lambda)-\frac{\mathrm{i}\epsilon\lambda}{4}(2C+\mathrm{i}\lambda)\,(1+A_4)\right.$$
$$\left.+\frac{\mathrm{i}}{2}\lambda(\epsilon-a_1)+\tfrac{1}{4}\lambda^2(1-2\epsilon)-\frac{\lambda^2 a_2}{8}+\frac{A}{c^2}\frac{4\lambda^2}{\pi\delta^3}\frac{x_c}{c}\right\}\alpha^0. \tag{83}$$

From the definition of A_4 it will be observed that it depends on a_1 and a_2, vanishing when these parameters vanish. Thus from the remark following equation (73) we see that the A_4 term in equation (83) represents the effect on the lift of allowing for the slowing up of the velocity of the wake vorticity due to the aerofoil thickness.

The coefficients of a_1 and a_2 in A_4 are given in table 1. Quadratic terms in a_1 and a_2 have been ignored in the preparation of this table. When the aerofoil degenerates to a flat plate satisfying the Joukowski condition, then $a_n = b_n = A = A_4 = 0$, $\delta = 1$, the equation (83) yields the standard result. (See, for example, Jones (1941).)

10. The moment

From (73) and (74)

$$c_3 = \frac{2H_3^{(2)}}{H_1^{(2)}+\mathrm{i}H_0^{(2)}}\left[\left\{\alpha^0\left(1+\frac{\mathrm{i}}{2}\lambda\right)-\frac{2\mathrm{i}\lambda y^0}{\delta c}\right\}(1-\epsilon)-\frac{\mathrm{i}}{2}\epsilon\lambda\alpha^0\right]\left(\frac{1-A_5}{1-A_1}\right)\mathrm{e}^{\mathrm{i}\nu t}, \tag{84}$$

where
$$A_5 = \lambda a_1\frac{H_2^{(2)}+\frac{1}{2}H_0^{(2)}}{H_3^{(2)}}+\tfrac{1}{4}(a_1-\mathrm{i}\lambda a_2)\frac{4H_1^{(2)}+3I_1^{(2)}}{4H_3^{(2)}},$$

but from (66),
$$\frac{H_3^{(2)}}{H_1^{(2)}+\mathrm{i}H_0^{(2)}} = \left(\frac{8}{\lambda^2}-\frac{4\mathrm{i}}{\lambda}-1\right)C+\frac{4\mathrm{i}}{\lambda},$$

so that from (65), (76) and (84)

$$c_3+\sigma_1-\sigma_2 = \frac{8}{\lambda}\mathrm{e}^{\mathrm{i}\nu t}\left[\left\{\alpha^0\left(1+\frac{\mathrm{i}}{2}\lambda\right)-\frac{2\mathrm{i}\lambda y^0}{\delta c}\right\}(1-\epsilon)-\frac{\mathrm{i}}{2}\epsilon\lambda\alpha^0\right]\left(\frac{2C}{\lambda}-\mathrm{i}C+\mathrm{i}\right)(1+A_6)-\mathrm{i}\lambda\alpha^0\,\mathrm{e}^{\mathrm{i}\nu t}, \tag{85}$$

where
$$A_6 = \frac{1}{1-A_1}\left\{A_1-A_5+\frac{\lambda^2 C(A_5-A_2)}{4(2C-\mathrm{i}\lambda C+\mathrm{i}\lambda)}\right\}.$$

The from (59), (63), (64), (81) and (85)

$$\frac{M}{\rho U^2 c^2\,\mathrm{e}^{\mathrm{i}\nu t}} = \frac{\pi}{8}\delta^2 b_1\,\mathrm{e}^{-\mathrm{i}\nu t}+\pi\delta\left\{-\frac{\mathrm{i}}{2}\lambda b_1+\frac{A}{c^2}\frac{4\lambda^2}{\pi\delta^3}\frac{y_c}{c}+\frac{b_2}{8}\lambda^2\right\}\frac{x^0}{c}$$
$$+\pi\delta\left\{\frac{\mathrm{i}}{2}(1-\epsilon)\,[(2\mathrm{i}C-\lambda)\,(1+A_4)-(2\mathrm{i}C-\lambda+C\lambda)\,(1+A_6)]\right.$$
$$\left.-\frac{\mathrm{i}}{2}\lambda(\epsilon-a_1)-\frac{\lambda^2 a_2}{8}+\frac{A}{c^2}\frac{4\lambda^2}{\pi\delta^3}\frac{x_c}{c}\right\}\frac{y^0}{c}$$
$$-\frac{\pi\delta^2}{8}\left\{2\left[\left(1+\frac{\mathrm{i}}{2}\lambda\right)-\epsilon(1+\mathrm{i}\lambda)\right]\left[\left(\frac{2\mathrm{i}C}{\lambda}-1\right)(1+A_4)-\left(\frac{2\mathrm{i}C}{\lambda}+C-1\right)(1+A_6)\right]\right.$$
$$\left.+\mathrm{i}\lambda(1-2\epsilon)-2\epsilon+2a_1+\tfrac{1}{4}\lambda^2(a_1+a_3)+\frac{32}{\pi}\frac{A}{c^2}\frac{\lambda^2}{\delta^4}\frac{k^2}{c^2}\right\}\alpha^0, \tag{86}$$

which agrees with the classical result when $\epsilon = 0$ and the aerofoil is a flat plate. The coefficients of a_1 and a_2 in A_6 are given in table 1.

11. Determination of the viscous slip and the centre of the profile from the steady aerofoil characteristics

Since $\lim_{\lambda=0} A_4, A_6 = 0$, and $\lim_{\lambda=0} C = 1$, in the case of steady flow (83) and (86) degenerate to

$$C_L = 2\pi\delta(1-\epsilon)\,\alpha^0,$$

and

$$C_M = \frac{\pi\delta^2}{2}(1-a_1)\,\alpha^0 + \frac{\pi\delta^2}{4}b_1,$$

where C_M is the moment coefficient about the profile centre. If $\tilde{C}_M$ is the moment coefficient about the point $(\tilde{x}, \tilde{y})$, then

$$\tilde{C}_M = C_M + \frac{\tilde{x}}{c}C_L.$$

These three equations yield

$$\frac{\partial C_L}{\partial \alpha} = 2\pi\delta(1-\epsilon), \tag{87}$$

$$\frac{\partial C_M}{\partial \alpha} = \frac{\pi\delta^2}{2}(1-a_1) + 2\pi\delta\frac{\tilde{x}}{c}(1-\epsilon). \tag{88}$$

When a_1 and δ have been calculated from the aerofoil co-ordinates,† these equations can be solved for ϵ and $\tilde{x}$ so that the left-hand sides of the equations have their experimental values.

Incidentally, equations (87) and (88) can be deduced directly from equations (16) and the steady form of Blasius's theorem (Woods 1953). This method yields the exact equations for the derivatives of C_L and C_M, but for the present application (87) and (88), which are correct to the first order in α, are quite sufficient. The number $1-\epsilon$ may be termed the 'Joukowski efficiency' of the aerofoil. For well-designed aerofoils this efficiency is between 80 and 90%.

Appendix 2 gives an equation for the position of the theoretical profile centre, which is usually between $0{\cdot}48c$ and $0{\cdot}50c$ from the leading edge for modern aerofoil shapes. The action of viscosity modifies the velocity distribution, particularly near the trailing edge, and this has the effect of displacing the profile centre to the rear of its theoretical position.

12. The air-load coefficients

Following Jones (1941) we shall denote the air-load coefficients by the real and imaginary parts of the non-dimensional numbers, l_{12}, l_{34}, m_{12} and m_{34}, defined by

$$\frac{L}{\rho U^2 c\, e^{i\nu t}} = l_{12}\frac{y^0}{c} + l_{34}\alpha^0,$$

and

$$\frac{M}{\rho U^2 c^2 e^{i\nu t}} = \frac{\pi}{8}\delta^2 b_1 e^{-i\nu t} + m_{12}\frac{y^0}{c} + m_{34}\alpha^0,$$

from which the comparatively unimportant displacement x^0 has been omitted. From (53), (54), (55), (83) and (86) it follows that

$$l_{12} = \pi\{(1-\epsilon)(1+A_4)(\lambda^2 - 2iC\lambda) + \lambda^2(\epsilon - a_1)\}, \tag{89}$$

$$l_{34} = \frac{\pi\delta}{2}\left\{(1+A_4)(2C+i\lambda)\left[(1-\epsilon)\left(1+\frac{i}{2}\lambda\right) - \frac{i\epsilon\lambda}{2}\right] + i\lambda(\epsilon - a_1) + \tfrac{1}{2}\lambda^2(1-2\epsilon) + \frac{2x_c}{\delta c}a_1\lambda^2\right\}, \tag{90}$$

† See equations (53) and (118).

$$m_{12} = \frac{\pi\delta}{2}\left\{i(1-\epsilon)[(2iC-\lambda)\,(1+A_4)-(2iC-\lambda+C\lambda)\,(1+A_6)]-i\lambda(\epsilon-a_1)+\frac{2x_c}{\delta c}a_1\lambda^2\right\}, \quad (91)$$

$$m_{34} = -\frac{\pi\delta^2}{8}\left\{2\left[\left(1+\frac{i\lambda}{2}\right)-\epsilon(1+i\lambda)\right]\left[\left(\frac{2iC}{\lambda}-1\right)(1+A_4)-\left(\frac{2iC}{\lambda}+C-1\right)(1+A_6)\right]\right.$$
$$\left.+i\lambda(1-2\epsilon)-2(\epsilon-a_1)+\frac{8k^2}{\delta^2c^2}a_1\lambda^2\right\}. \quad (92)$$

If $l_{12} = l_1+il_2$, $m_{12} = m_1+im_2$, etc., where l_1, l_2, m_1, m_2, etc., are real numbers, then to first order in ϵ, A_4 and A_6 these equations can be written

$$\left.\begin{aligned}
l_1 &= l_{10}(1+a_1L_1+\epsilon N_1), & l_2 &= l_{20}(1+a_1L_2+\epsilon N_2),\\
l_3 &= \delta l_{30}(1+a_1L_3+\epsilon N_3)+\frac{\pi a_1 x_c}{c}\lambda^2, & l_4 &= \delta l_{40}(1+a_1L_4+\epsilon N_4),\\
m_1 &= \delta m_{10}(1+a_1M_1+\epsilon Q_1)+\frac{\pi a_1 x_c}{c}\lambda^2, & m_2 &= \delta m_{20}(1+a_1M_2+\epsilon Q_2),\\
m_3 &= \delta^2 m_{30}(1+a_1M_3+\epsilon Q_3)-\frac{\pi a_1 k^2}{c^2}\lambda^2, & m_4 &= \delta^2 m_{40}(1+a_1M_4+\epsilon Q_4),
\end{aligned}\right\} \quad (93)$$

where

$$l_{10} = \pi\lambda(\lambda-2B), \qquad l_{20} = 2\pi A\lambda,$$
$$l_{30} = \tfrac{1}{2}\pi(2A+\lambda B), \qquad l_{40} = \tfrac{1}{2}\pi(\lambda A+\lambda-2B),$$
$$m_{10} = -\tfrac{1}{2}\pi B\lambda, \qquad m_{20} = -\tfrac{1}{2}\pi A\lambda,$$
$$m_{30} = \frac{\pi}{8}(2A+B\lambda), \qquad m_{40} = -\frac{\pi}{8}(2B+\lambda-\lambda A),$$
$$L_1 = \frac{\mathscr{R}[A_4(2iC-\lambda)+a_1\lambda]}{a_1(2B-\lambda)}, \qquad N_1 = \frac{-2B}{2B-\lambda},$$
$$L_2 = \frac{\mathscr{I}[A_4(2iC-\lambda)]}{2a_1A}, \qquad N_2 = -1,$$
$$L_3 = \frac{\mathscr{R}\left[(2C+i\lambda)\left(1+\frac{i}{2}\lambda\right)A_4\right]}{a_1(2A+\lambda B)}, \qquad N_3 = -\frac{2(A+\lambda B)}{(2A+\lambda B)},$$
$$L_4 = \frac{\mathscr{I}\left[(2C+i\lambda)\left(1+\frac{i}{2}\lambda\right)A_4-ia_1\lambda\right]}{a_1(\lambda A+\lambda-2B)}, \qquad N_4 = -\frac{2(\lambda A-B)}{(\lambda A+\lambda-2B)},$$
$$M_1 = \frac{\mathscr{R}[(2C+i\lambda)\,A_4-(2C+i\lambda-iC\lambda)\,A_6]}{a_1\lambda B}, \qquad Q_1 = -1,$$
$$M_2 = \frac{\mathscr{I}[(2C+i\lambda)\,A_4-(2C+i\lambda+iC\lambda)\,A_6-i\lambda a_1]}{a_1\lambda A}, \qquad Q_2 = \frac{1-A}{A},$$
$$M_3 = -\frac{2\mathscr{R}\left[\left(1+\frac{i}{2}\lambda\right)\{(2iC-\lambda)\,A_4-(2iC+C\lambda-\lambda)\,A_6\}+a_1\lambda\right]}{a_1\lambda(2A+B\lambda)}, \qquad Q_3 = \frac{2(1-A-B\lambda)}{2A+B\lambda},$$
$$M_4 = \frac{2\mathscr{I}\left[\left(1+\frac{i}{2}\lambda\right)\{(2iC-\lambda)\,A_4-(2iC+C\lambda-\lambda)\,A_6\}\right]}{a_1\lambda(\lambda+2B-\lambda A)}, \qquad Q_4 = -\frac{2(\lambda A-B-\lambda)}{(\lambda A-\lambda-2B)},$$

in which '$\mathscr{I}$' denotes the 'imaginary part of', and A and $-B$ are the real and imaginary parts of Theodorsen's function, C. The twenty-four functions of λ defined above are set out in tables 3 and 4. The functions l_{10}, l_{20}, ..., m_{10}, m_{20}, ... are the same as those occurring in the classical flat plate theory for the air-load coefficients l_1, l_2, ..., m_1, m_2, ...

Table 3. The air-load 'stiffness' coefficients

λ	l_{10}	L_1	N_1	l_{30}	L_3	N_3
0	0	0	−1·0000	3·1416	0	−1·0000
0·02	−0·0082	0·2783	−1·1534	3·0300	0·0063	−1·0008
0·04	−0·0241	0·3761	−1·2083	2·9186	0·0102	−1·0025
0·06	−0·0425	0·4704	−1·2664	2·8159	0·0126	−1·0048
0·08	−0·0605	0·5706	−1·3322	2·7233	0·0138	−1·0074
0·1	−0·0768	0·6814	−1·4088	2·6406	0·0141	−1·0103
0·2	−0·1114	1·6141	−2·1284	2·3450	0·0071	−1·0252
0·3	−0·0553	6·4048	−6·1161	2·1736	−0·0093	−1·0389
0·4	0·0880	−6·3986	4·7117	2·0671	−0·0344	−1·0501
0·5	0·3119	−2·5727	1·5179	1·9968	−0·0697	−1·0593
0·6	0·6115	−1·7522	0·8494	1·9482	−0·1164	−1·0667
0·7	0·9834	−1·3961	0·5654	1·9132	−0·1752	−1·0726
0·8	1·4250	−1·2014	0·4109	1·8873	−0·2459	−1·0776
0·9	1·9348	−1·0840	0·3152	1·8676	−0·3274	−1·0816
1·0	2·5116	−1·0111	0·2509	1·8522	−0·4181	−1·0850

λ	m_{10}	M_1	Q_1	m_{30}	M_3	Q_3
0	0	—	−1·0000	0·7854	—	0
0·02	−0·0024	−0·5700	−1·0000	0·7575	−0·9920	0·0361
0·04	−0·0073	−0·5836	−1·0000	0·7297	−1·0078	0·0739
0·06	−0·0134	−0·5895	−1·0000	0·7040	−1·0129	0·1109
0·08	−0·0202	−0·5972	−1·0000	0·6808	−1·0180	0·1462
0·1	−0·0271	−0·6056	−1·0000	0·6602	−1·0227	0·1795
0·2	−0·0593	−0·6740	−1·0000	0·5863	−1·0375	0·3144
0·3	−0·0845	−0·7875	−1·0000	0·5434	−1·0423	0·4065
0·4	−0·1037	−0·9491	−1·0000	0·5168	−1·0446	0·4697
0·5	−0·1184	−1·1604	−1·0000	0·4992	−1·0506	0·5140
0·6	−0·1299	−1·4237	−1·0000	0·4871	−1·0639	0·5459
0·7	−0·1390	−1·7389	−1·0000	0·4783	−1·0693	0·5694
0·8	−0·1464	−2·1039	−1·0000	0·4718	−1·1222	0·5870
0·9	−0·1525	−2·5195	−1·0000	0·4669	−1·1692	0·6006
1·0	−0·1575	−2·9789	−1·0000	0·4630	−1·2284	0·6111

It should be noticed that the flat plate results are modified by aerofoil thickness in three ways:

(1) the parameter $\frac{1}{2}\omega$ is replaced by λ $(=\frac{1}{2}\omega\delta)$,

(2) l_{34} and m_{12} are multiplied by δ, while m_{34} is multiplied by δ^2,

(3) the contributions from the a_1 terms in (93).

The effects of viscosity are represented in the theory in two ways, namely,

(1) in the displacement of the profile centre to the rear of its theoretical position, which modifies the value of $\tilde{x}$ occurring in equations (94) below, and

(2) in introducing the ϵ term in (93).

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By transforming the moments and displacements to an axis at $(\tilde{x}, 0)$ it is found that the air-load coefficients transform according to the formulae

$$\left.\begin{aligned} \tilde{l}_{12} &= l_{12}, & \tilde{l}_{34} &= l_{34}+\frac{\tilde{x}}{c}l_{12}, \\ \tilde{m}_{12} &= m_{12}+\frac{\tilde{x}}{c}l_{12}, & \tilde{m}_{34} &= m_{34}+\frac{\tilde{x}}{c}(l_{34}+m_{12})+\left(\frac{\tilde{x}}{c}\right)^2 l_{12}. \end{aligned}\right\} \tag{94}$$

TABLE 4. THE AIR-LOAD 'DAMPING' COEFFICIENTS

λ	l_{20}	L_2	N_2	l_{40}	L_4	N_4
0	0	0	−1·0000	0	0	−1·0000
0·02	−0·1202	0·0062	−1·0000	−0·1746	0·3255	−1·0065
0·04	−0·2329	0·0099	−1·0000	−0·2434	0·4637	−1·0189
0·06	−0·3363	0·0119	−1·0000	−0·2697	0·6134	−1·0377
0·08	−0·4325	0·0126	−1·0000	−0·2701	0·7940	−1·0649
0·1	−0·5227	0·0123	−1·0000	−0·2535	1·0263	−1·0763
0·2	−0·9143	0·0014	−1·0000	−0·0498	9·0096	−2·7170
0·3	−1·2534	−0·0182	−1·0000	0·2213	−2·6921	−0·2864
0·4	−1·5707	−0·0439	−1·0000	0·5027	−1·4345	−0·5313
0·5	−1·8785	−0·0763	−1·0000	0·7815	−1·0727	−0·5960
0·6	−2·1820	−0·1166	−1·0000	1·0551	−0·9066	−0·6238
0·7	−2·4839	−0·1655	−1·0000	1·3234	−0·8186	−0·6384
0·8	−2·7854	−0·2237	−1·0000	1·5870	−0·7724	−0·6470
0·9	−3·0872	−0·2913	−1·0000	1·8467	−0·7527	−0·6524
1·0	−3·3894	−0·3681	−1·0000	2·1031	−0·7521	−0·6560

λ	m_{20}	M_2	Q_2	m_{40}	M_4	Q_4
0	0	−1·0000	0	0	—	−1·0000
0·02	−0·0303	—	0·0377	−0·0594	−0·5713	−1·0048
0·04	−0·0587	−1·0088	0·0791	−0·0923	−0·5874	−1·0125
0·06	−0·0853	−1·0150	0·1210	−0·1145	−0·5962	−1·0222
0·08	−0·1081	−1·0212	0·1622	−0·1304	−0·6063	−1·0336
0·1	−0·1307	−1·0270	0·2020	−0·1419	−0·6177	−1·0465
0·2	−0·2286	−1·0531	0·3744	−0·1695	−0·6993	−1·1262
0·3	−0·3134	−1·0526	0·5038	−0·1803	−0·8112	−1·2189
0·4	−0·3927	−1·0497	0·6001	−0·1885	−0·9390	−1·3125
0·5	−0·4696	−1·0436	0·6724	−0·1973	−1·0703	−1·4001
0·6	−0·5455	−1·0383	0·7277	−0·2075	−1·1959	−1·4784
0·7	−0·6210	−1·0363	0·7707	−0·2189	−1·3093	−1·5465
0·8	−0·6964	−1·0396	0·8046	−0·2316	−1·4069	−1·6049
0·9	−0·7718	−1·0491	0·8317	−0·2452	−1·4863	−1·6545
1·0	−0·8473	−1·0658	0·8538	−0·2596	−1·5466	−1·6966

13. THE AIR-LOAD COEFFICIENTS IN STILL AIR

From (63), (89) to (92) and (94)

$$\left.\begin{aligned} &\lim_{U=0}\frac{\tilde{l}_{12}}{(\omega/2)^2} = \delta^2\pi(1-a_1), \\ &\lim_{U=0}\frac{\tilde{l}_{34}}{(\omega/2)^2} = \lim_{U=0}\frac{\tilde{m}_{12}}{(\omega/2)^2} = \delta^2\pi\left\{\frac{\tilde{x}}{c}+a_1\left(\frac{x_c}{c}-\frac{\tilde{x}}{c}\right)\right\}, \\ &\lim_{U=0}\frac{\tilde{m}_{34}}{(\omega/2)^2} = \delta^2\pi\left\{\left(\frac{\tilde{x}}{c}\right)^2 - a_1\left[\left(\frac{\tilde{x}}{c}\right)^2 - 2\left(\frac{x_c}{c}\right)+\frac{k^2}{c^2}\right]\right\}, \end{aligned}\right\} \tag{95}$$

so that the theory yields no still air damping. However, viscous damping of the type that is present in still air does make a significant contribution to l_2, and a negligible contribution to the other air-load coefficients (about the origin). For example, from some experimental results given by Berg (1952) it is found that in still air

$$\frac{l_2}{(\frac{1}{2}\omega)^2} = -0\cdot146\pi, \quad \frac{l_4}{(\frac{1}{2}\omega)^2} = \frac{m_2}{(\frac{1}{2}\omega)^2} = 0, \quad \frac{m_4}{(\frac{1}{2}\omega)^2} = -0\cdot008\pi.$$

To the author's knowledge no adequate theory exists to predict the viscous damping on an aerofoil in still and moving air. Since it is likely that at low values of U, i.e. at high values of λ, the viscous damping has nearly its still-air value, then under these conditions the author's theory can be expected to yield values of the damping coefficients relative to still air. These relative coefficients are often obtained as a first step by experimentalists in the determination of the absolute values of the coefficients. Some experimental evidence appears in the next section on this point.

14. Comparison with experiment

The most recent experimental determination of the air-load coefficients in incompressible two-dimensional flow was made at the Nationaal Luchtvaarlaboratorium, Amsterdam, by Berg, Van der Vooren and others. While the experiments and subsequent reduction of the data appear to have been made with considerable care the reports describing the work give only two facts about the profile used, namely, that it is 7·3 % thick, with this maximum thickness occurring at $0\cdot30c$ from the leading edge. This omission is not surprising when it is realized that, although only one aerofoil was used in these experiments, it is claimed that the final results are applicable to all conventional aerofoil shapes. In as much as conventional aerofoil shapes are *about* the same thickness and have *about* the same Joukowski efficiency it is certainly true that the experimental results summarized by Berg & Van der Vooren (1952) are an improvement over the theoretical flat plate results, but no more than this can be concluded without conducting experiments in which the aerofoil thickness and shape are varied. The work of Bratt & Wight (1945) indicates that the air loads do vary significantly with aerofoil shape.

From the data given about the aerofoil profile, and experience with aerofoils with similar characteristics the author estimates that for the N.L.L. aerofoil, $a_1 \approx 0\cdot12$, $\delta \approx 1\cdot058$, and $x_c \approx 0\cdot35c$ from the leading edge. From the results in figures 3 and 5 of the paper by Berg & Van der Vooren (1952) we find

$$\frac{1}{2\pi}\frac{\partial C_L}{\partial \alpha} = 0\cdot875, \quad \frac{\partial \tilde{C}_m}{\partial \alpha} = 0,$$

where $\tilde{C}_M$ is about the quarter-chord point. When these values are substituted in equations (87) and (88) it is found that $\epsilon = 0\cdot17$ and that the profile centre is at $0\cdot53c$ from the leading edge. Thus the quarter-chord point is at $\tilde{x} = -0\cdot28c$, $\tilde{x}$ being measured from the profile centre. Using these values of δ, a_1, ϵ and $\tilde{x}/c$ in equations (93) and (94) we obtain finally the damping air-load coefficients shown by the continuous lines in figures 6 and 7.

The damping relative to still air shown in these figures was obtained by subtracting Berg's (1952) experimental values of the still air damping from the experimental values of the air loads given by Berg & Van der Vooren (1952).† Viscous damping is probably responsible for the discrepancy between theory and experiment in the case of $\tilde{l}_2$, but it will be noticed that the theory agrees quite well with the relative damping when ω is large, a result that was anticipated in the previous section. The agreement between experiment and theory is good for $\tilde{m}_4$ and $(\tilde{m}_2+\tilde{l}_4)$, but poor for $\tilde{l}_4$ and $\tilde{m}_2$ at high values of ω. This may not

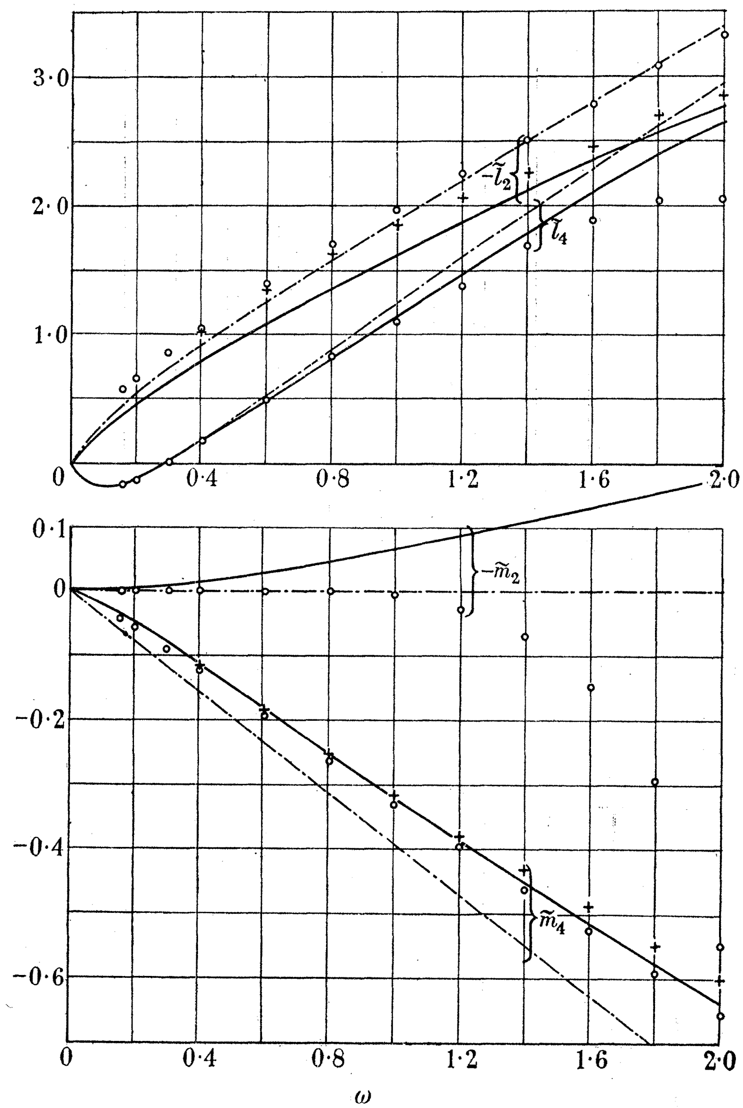

FIGURE 6. The damping air-load coefficients (axis at quarter-chord point). –·–·–· flat plate theory; — author's theory; ○ experiment, absolute values; + experiment, relative to still air (from Berg (1952) and Berg & Van de Vooren (1952)).

be coincidental for the experiments were of two types, namely, (i) with fixed axes of rotation, which enabled $\tilde{l}_2$, $(\tilde{m}_2+\tilde{l}_4)$ and $\tilde{m}_4$‡ to be derived, and (ii) with oscillating axes of rotation, enabling $\tilde{m}_2$ and $\tilde{l}_4$ to be separated. Thus the author's theory is in good agreement with the results of the first type of experiment but not with the second.

† The air-load coefficients given by Berg & Van de Vooren are related to those defined above by

$$\tilde{l}_{12} = -\pi k_a, \quad \tilde{l}_{34} = \frac{\pi}{2} k_b, \quad \tilde{m}_{12} = \frac{\pi}{2} m_a, \quad \tilde{m}_{34} = -\frac{\pi}{4} m_b.$$

‡ Moments only were measured; cf. the last of equations (94).

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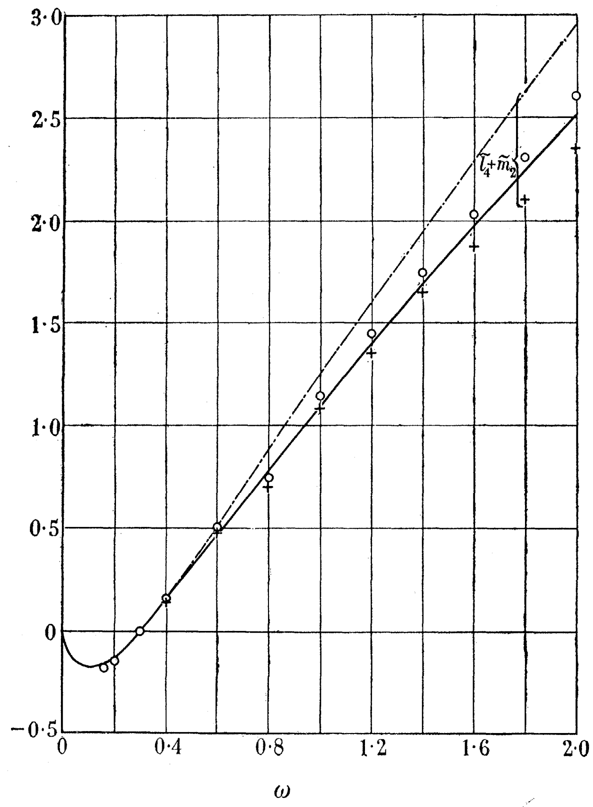

FIGURE 7. Values of $\tilde{m}_2+l_4$ (axis at quarter-chord point). —·—·— flat plate theory; — author's theory; ○ experiment, absolute values; + experiment, relative to still air (from Berg (1952) and Berg *et al.* (1952)).

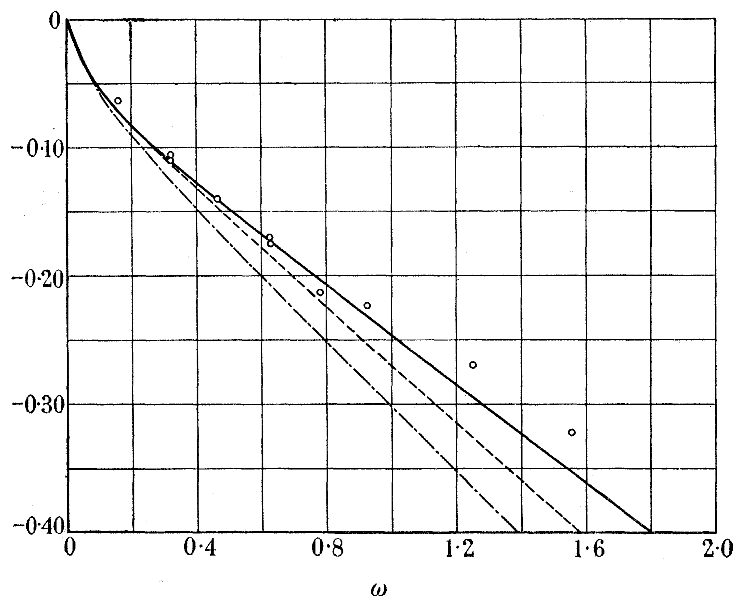

FIGURE 8. Values of $\tilde{m}_4$ about the third-chord axis of a 15% thick Joukowski aerofoil. —·—·— flat plate theory; ---- W. P. Jones's theory; ——— author's theory; ○ experiment (from Bratt & Wight (1945)).

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The author's theory yields values of the air-load stiffness coefficients, $\tilde{l}_1$ and $\tilde{l}_3$, which are in poor agreement with the N.L.L. experimental values in the range $1{\cdot}0 \leqslant \omega \leqslant 2{\cdot}0$. It is possible that the agreement could be improved by introducing a phase lag into the oscillation of the rear stagnation point, but further experimental evidence is desirable before doing this. However the still air stiffness (or inertia) coefficients given by Berg (1952) tend to confirm equation (95) as shown in the following table:

	$\tilde{l}_1/(\frac{1}{2}\omega)^2$	$l_3/(\frac{1}{2}\omega)^2$	$\tilde{m}_1/(\frac{1}{2}\omega)^2$	$\tilde{m}_3/(\frac{1}{2}\omega)^2$
mean expt. value	3·08	0·87	0·87	0·25
equation (95)	3·08	0·94	0·94	0·25
flat-plate theory	3·142	0·785	0·785	0·197

Some further experimental evidence supporting the present theory is shown in figure 8. The experimental results were taken from Bratt & Wight (1945), while the curve due to W. P. Jones was taken from his 1948 paper (see comments in the Introduction). For the 15% thick Joukowski aerofoil used in the experiments it is found that $\delta \approx 1{\cdot}12$ and $a_1 \approx 0{\cdot}20$; then from the steady-flow results given by Bratt & Wight (1945) (C_M against α for two axis positions) and equation (88) we find that $\epsilon = 0{\cdot}12$, and that the profile centre is at the mid-chord point. An application of table 4 and equation (94) then yields the theoretical curve shown in figure 8.

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Appendix 1. Calculation of the aerofoil parameters

The integrals appearing in equations (26), (27), (46), (49) and (50), which depend only on the aerofoil shape, can be reduced to simple forms by using the equation (Woods 1950)

$$Uz_0 = Uz_a + w_0 - \frac{\mathrm{i}U}{2\pi}\int_{-\pi}^{\pi} y_0(\gamma^*) \coth \tfrac{1}{2}(\zeta - \mathrm{i}\gamma^*)\, \mathrm{d}\gamma^*. \qquad (101)$$

The proof of this equation is similar to that of equation (16) given in § 2. On examining the proof of equation (16) again, it will be noticed that it holds for *any* analytic function f_0 satisfying $\lim_{w_0=\infty} f_0(w_0) = 0$, with its imaginary part θ_0 specified on the aerofoil boundary. The function

$$f_0 = Uz_0 - w_0 - Uz_a$$

is clearly an analytic function in the z_0, w_0 and ζ planes, and furthermore it follows from (36) that $\lim_{w_0=\infty} f_0(w_0) = 0$. On the aerofoil boundary ψ_0 vanishes so that the imaginary part of f_0 is $U(y_0 - y_a)$. Thus we have by comparison with equation (16),

$$Uz_0 = Uz_a + w_0 + \frac{U}{2\pi}\int_{-\pi}^{\pi} (y_0 - y_a) \cot \tfrac{1}{2}(\gamma^* + \mathrm{i}\zeta)\, \mathrm{d}\gamma^*,$$

from which equation (101) immediately follows.

Differentiating equation (101) with respect to w_0, and making use of equation (20) we find

$$U\frac{\mathrm{d}z_0}{\mathrm{d}w_0} = 1 - \frac{\mathrm{i}U}{8a\pi}\int_{-\pi}^{\pi} y_0 \operatorname{cosech}\zeta \operatorname{cosech}^2 \tfrac{1}{2}(\zeta - \mathrm{i}\gamma^*)\, \mathrm{d}\gamma^*$$

$$= 1 + \frac{\mathrm{i}U}{a\pi}\left(\frac{a}{w_0}\right)^2 \int_{-\pi}^{\pi} y_0 \,\mathrm{e}^{-\mathrm{i}\gamma}\, \mathrm{d}\gamma - \frac{2\mathrm{i}U}{a\pi}\left(\frac{a}{w_0}\right)^3 \int_{-\pi}^{\pi} y_0\, \mathrm{e}^{-2\mathrm{i}\gamma}\, \mathrm{d}\gamma$$

$$+ \frac{3\mathrm{i}U}{a\pi}\left(\frac{a}{w_0}\right)^4 \int_{-\pi}^{\pi} y_0 (\mathrm{e}^{-\mathrm{i}\gamma} + \mathrm{e}^{-3\mathrm{i}\gamma})\, \mathrm{d}\gamma + O\left(\frac{a}{w_0}\right)^5. \tag{102}$$

Comparison with equation (25) yields

$$B_0 = \frac{\mathrm{i}U}{a\pi}\int_{-\pi}^{\pi} y_0\, \mathrm{e}^{-\mathrm{i}\gamma}\, \mathrm{d}\gamma = a_1 + \mathrm{i}b_1, \tag{103}$$

and

$$C_0 = -\frac{2\mathrm{i}U}{a\pi}\int_{-\pi}^{\pi} y_0\, \mathrm{e}^{-2\mathrm{i}\gamma}\, \mathrm{d}\gamma = -2(a_2 + \mathrm{i}b_2), \tag{104}$$

where a_n and b_n $(n = 1, 2)$ are defined by equations (51) and (52) respectively.

Now the integrals

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{U\cos\theta_0}{q_0} \sin m\gamma\, \mathrm{d}\gamma, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{U\sin\theta_0}{q_0} \sin m\gamma\, \mathrm{d}\gamma$$

$(m = 1, 2, 3)$, occurring in equations (47), (49) and (50), are the real and imaginary parts of I_m, where

$$I_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{U\mathrm{e}^{\mathrm{i}\theta_0}}{q_0} \sin m\gamma\, \mathrm{d}\gamma = -\frac{1}{\pi}\int_{-\pi}^{\pi} U\frac{\mathrm{d}z_0}{\mathrm{d}w_0} \sinh m\zeta\, \mathrm{d}\zeta,$$

since $\zeta = \mathrm{i}\gamma$ on the aerofoil. Thus from equation (6)

$$I_m = -\frac{1}{2aU}\int_C U\frac{\mathrm{d}z_0}{\mathrm{d}w_0}\frac{\sinh m\zeta}{\sinh\zeta}\, \mathrm{d}w_0,$$

where, since the integrand is analytic, C is any contour enclosing the aerofoil, taken in an anti-clockwise direction. From equations (20), (102), (103) and (104) it follows with the aid of the theorem of residues that

$$I_1 = 0, \quad I_2 = -b_1 + \mathrm{i}a_1, \quad I_3 = -2b_2 + 2\mathrm{i}a_2. \tag{105}$$

Similarly if
$$J_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{U e^{i\theta_0}}{q_0} \cos m\gamma \, d\gamma \quad (m = 0, 1, 2, 3),$$
then
$$J_0 = 2, \quad J_1 = 0, \quad J_2 = a_1 + ib_1, \quad J_3 = 2a_2 + 2ib_2. \tag{106}$$

Now consider integrals of the type
$$K_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{U}{q_0}(x_0 \cos\theta_0 + y_0 \sin\theta_0) \sin m\gamma \, d\gamma \quad (m = 1, 2, 3),$$
i.e.
$$K_m = \mathscr{R}\frac{1}{2a\pi}\int_{-\pi}^{\pi} \bar{z}_0 U \frac{dz_0}{dw_0} \frac{\sinh m\zeta}{\sinh\zeta} dw_0. \tag{107}$$
Clearly
$$K_1 = 0. \tag{108}$$
The integrand in (107) is not analytic, and so the integral cannot be evaluated by the theorem of residues, but if $y_0 \, dy_0$—a second order term in thickness—is neglected,† then $\mathscr{R}\bar{z}_0 dz_0 = \mathscr{R}z_0 dz_0$, so that (107) becomes
$$K_m = -\mathscr{R}\frac{1}{2a\pi}\int_C z_0 U \frac{dz_0}{dw_0} \frac{\sinh m\zeta}{\sinh\zeta} dw_0.$$
From (25), (36), (102) and the theorem of residues it follows that
$$K_2 = \frac{ab_2}{U}, \quad K_3 = \frac{2a}{U}(b_1 + b_3 - a_1 b_1), \tag{109}$$
where the origin has been taken at the centre of the profile as in § 5.

Making a similar approximation we find that if
$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{U}{q_0}(x_0 \cos\theta_0 + y_0 \sin\theta_0) \cos m\gamma \, d\gamma \quad (m = 0, 1, 2, 3),$$
then
$$\left.\begin{aligned} L_0 = 0, \qquad L_1 = -\frac{2a}{U}, \\ L_2 = -\frac{aa_2}{U}, \quad L_3 = -\frac{a}{U}(2a_1 + 2a_3 - a_1^2 + b_1^2). \end{aligned}\right\} \tag{110}$$

The coefficients b_n vanish for a symmetrical aerofoil; thus they represent the effect of aerofoil camber. For modern aerofoils they are much smaller than the coefficients a_n.

Useful approximations for a_n can be found by making use of the approximation
$$Ux_0 = -2a \cos\gamma.$$
The equation for a_1 becomes
$$a_1 = \frac{U}{a\pi}\int_{-\pi}^{\pi} y_0 \sin\gamma \, d\gamma \approx \left(\frac{Uc}{4a}\right)^2 \frac{8}{\pi}\int_{-\pi}^{\pi} y_0 \, dx_0,$$
i.e.
$$a_1 = \frac{8}{\pi}\frac{A}{\delta^2 c^2}, \tag{111}$$
where
$$\delta = \left(\frac{4a}{Uc}\right), \tag{112}$$

† The same degree of approximation has already been used in deriving equation (43).

and A is the profile area. The number δ is discussed in appendix 2. Similarly

$$a_2 \approx -\frac{4x_c}{c} a_1, \tag{113}$$

and

$$a_3 \approx \left(\frac{16k^2}{\delta^2 c^2} - 1\right) a_1, \tag{114}$$

where x_c is the position of the centroid, and k is the radius of gyration about the profile centre.

Appendix 2. The values of δ and x_a/c

On the aerofoil surface, $\eta = 0$, equation (101) becomes

$$Ux_0 = Ux_a + \phi_0 - \frac{U}{2\pi}\int_{-\pi}^{\pi} y_0(\gamma^*) \cot \tfrac{1}{2}(\gamma - \gamma^*)\, d\gamma^*, \tag{115}$$

where the origin of x_0 is here taken to be at the leading edge, $\gamma = 0$, or $\phi_0 = -2a$. At this leading edge (115) becomes

$$O = Ux_a - 2a + \frac{U}{2\pi}\int_{-\pi}^{\pi} y_0(\gamma^*) \cot \tfrac{1}{2}\gamma^*\, d\gamma^*,$$

while at the trailing edge, $\gamma = \pi$, $\phi_0 = 2a$, $x_0 = c$, it yields

$$U_c = Ux_a + 2a - \frac{U}{2\pi}\int_{-\pi}^{\pi} y_0(\gamma^*) \tan \tfrac{1}{2}\gamma^*\, d\gamma^*.$$

Addition and subtraction of these results yields

$$\delta = 1 + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{y_0}{c}\frac{1}{\sin\gamma}\, d\gamma, \tag{116}$$

and

$$\frac{x_a}{c} = \frac{1}{2}\left\{1 - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{y_0}{c}\cot\gamma\, d\gamma\right\}. \tag{117}$$

If the approximation $\dfrac{2x_0}{\delta c} = 1 - \cos\gamma$ is used, these equations become

$$\frac{1}{\delta} = 1 - \frac{1}{2\pi}\int_0^c (y_u - y_l)\frac{dx}{x(c-x)}, \tag{118}$$

and

$$\frac{x_a}{c} = \frac{1}{2}\left\{1 - \frac{1}{2\pi c}\int_0^c (y_u - y_l)\,\frac{c-2x}{x(c-x)}\, dx\right\}, \tag{119}$$

where y_u and y_l denote the values of y on the upper and lower surfaces respectively.